Quantum Mechanics
Lecture Notes Part 2

The predictions of Q.M. are deeply mysterious.
- behaviour when measured (cf. Stern-Gerlach)
- quantum superpositions
- quantum entanglement / quantum non-locality

What does it all mean?
↳ this part of the course

Reminder:

Postulates of Q.M.

1. The state space of a quantum system is a Hilbert space, and individual states are normalised vectors in that space.

2. The evolution of a quantum system is described by the Schrödinger equation.

3. Measurements of a quantum system are described by self-adjoint operators A on its Hilbert space.
   - Outcomes: eigenvalues λ_i
   - Outcome probability: $|\langle e_i | \psi \rangle|^2$ (eigvect $|e_i\rangle$)
   - Post-measurement state: $|e_i\rangle$ (given outcome λ_i)

Health warning!

Postulate 3 predicts a discontinuous evolution when system is measured (wavefunction "collapses" into an eigenstate).

But the measurement apparatus (and ultimately even the laboratory and the experimenter) are made of particles obeying Q.M., i.e. undergoing a continuous evolution (Postulate 2).

Postulates 2 & 3 contradict each other! (The "measurement problem".)

This is the elephant in the closet in Q.M. and we still don't have a good explanation.

What does it all mean?
↳ We still don't have a complete answer.

But we know some things, and they're both fascinating and surprising...

Lectures

I 1. Measurement

2. Tensor products
3. & multiple particles

4. ERR, Bell inequalities
5. & non-locality

I 6. (bonus topic?!)

Bibliography

No single ideal text book:

Nielsen & Chuang (Chapter 2)

Wolf & Werner review paper
www.arXiv.org/abs/quant-ph/2 0107093

John Bell "Speakable & Unspeakable in Q.M."

I. Measurement

Review of Dirac Notation

Given two states $|\psi\rangle$, $|\varphi\rangle$, we can make an operator

$$|\psi\rangle\langle\varphi|$$

(technically, $\langle\varphi|$ lives in the dual space $\mathcal{H}^*$ of linear forms on original Hilbert space $\mathcal{H}$)

which acts on a state $|\phi\rangle$ as

$$(|\psi\rangle\langle\varphi|)\,|\phi\rangle = \underbrace{|\psi\rangle}_{\text{state}}\ \underbrace{\langle\varphi|\phi\rangle}_{\text{complex number}}$$

In particular, we have projector

$$P_\psi = |\psi\rangle\langle\psi|$$

This gives a convenient notation for self-adjoint operators.

E.g. let $A \in \mathbb{C}^2$ have eigvals λ_1, λ_2 and corresponding eigvects. $|e_1\rangle$, $|e_2\rangle$. Then

$$A = \lambda_1 |e_1\rangle\langle e_1| + \lambda_2 |e_2\rangle\langle e_2|$$

Check:

$$A|e_1\rangle = (\lambda_1 |e_1\rangle\langle e_1|)|e_1\rangle + (\lambda_2 |e_2\rangle\langle e_2|)|e_2\rangle$$

$$= \lambda_1 |e_1\rangle (\underbrace{\langle e_1|e_1\rangle}_{=1}) + \lambda_2 |e_2\rangle (\underbrace{\langle e_2|e_1\rangle}_{=0})$$

$$= \lambda_1 |e_1\rangle$$

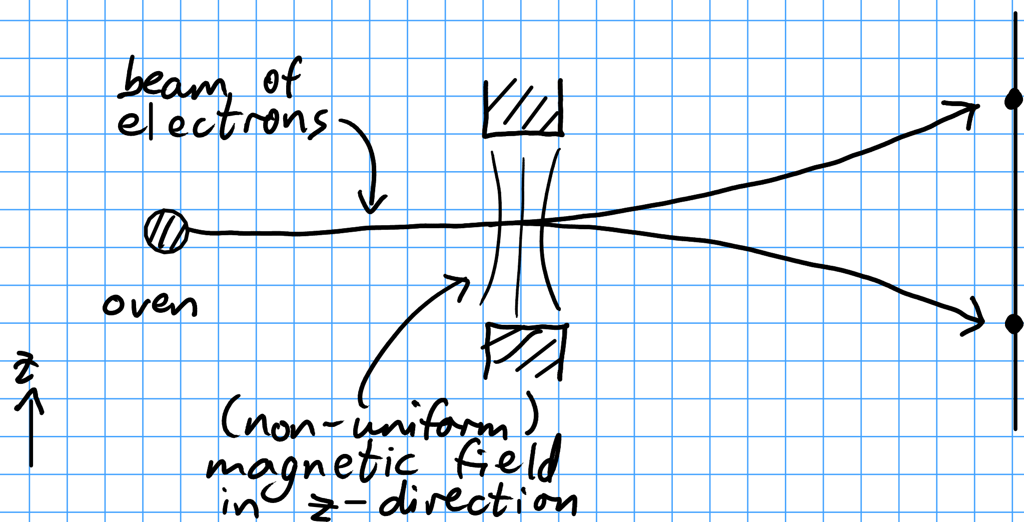
as required

eigvects of self-adj. op. are orthogonal

Measurement

We will review measurement in Q.M. by explaining the Stern-Gerlach experiment.

Example 1: basic S-G experiment



Why only two beams, nothing in between?

Electrons have spin $-\frac{1}{2}$

- state space spanned by $|+\frac{1}{2}\rangle$, $|-\frac{1}{2}\rangle$.
- x, y, z components of spin given by
  $J_1 = \frac{\hbar}{2} X$, $J_2 = \frac{\hbar}{2} Y$, $J_3 = \frac{\hbar}{2} Z$.

[Recall Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where state $\alpha|\frac{1}{2}\rangle + \beta|-\frac{1}{2}\rangle \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$]

Note that state space contains <u>all</u> orientations of spin (not just z-direction):

E.g. spin aligned along $+x$ direction is state

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\rangle + |-\tfrac{1}{2}\rangle\right)$$

Check:

$$J_1|+\rangle = \frac{1}{2}(J_+ + J_-)\frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\rangle + |-\tfrac{1}{2}\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\overset{=0}{\cancel{J_+|\tfrac{1}{2}\rangle}} + J_-|-\tfrac{1}{2}\rangle + J_-|\tfrac{1}{2}\rangle + \overset{=0}{\cancel{J_-|-\tfrac{1}{2}\rangle}}\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\hbar|\tfrac{1}{2}\rangle + \hbar|-\tfrac{1}{2}\rangle\right)$$

$$= \frac{\hbar}{2}|+\rangle \quad \text{as claimed.}$$

Electrons produced with randomly oriented spins:

$$|\psi\rangle = \alpha|\tfrac{1}{2}\rangle + \beta|-\tfrac{1}{2}\rangle$$

α, β random (subject to normalisation $|\alpha|^2 + |\beta|^2 = 1$)

S-G apparatus measures z-component J_3:

$$J_3 = \frac{\hbar}{2}|\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| - \frac{\hbar}{2}|-\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}|$$

eig vals.

projectors onto eig vects.

There are only two possible outcomes, given by eig vals. $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

Probability of outcome $\frac{\hbar}{2}$:

$$p(\tfrac{\hbar}{2}) = \langle\psi| \left(|\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| \right) |\psi\rangle$$

$$= \left(\alpha^* \langle\tfrac{1}{2}| + \beta^* \langle -\tfrac{1}{2}|\right) |\tfrac{1}{2}\rangle\langle\tfrac{1}{2}| \left(\alpha |\tfrac{1}{2}\rangle + \beta |-\tfrac{1}{2}\rangle\right)$$

note complex conjugates

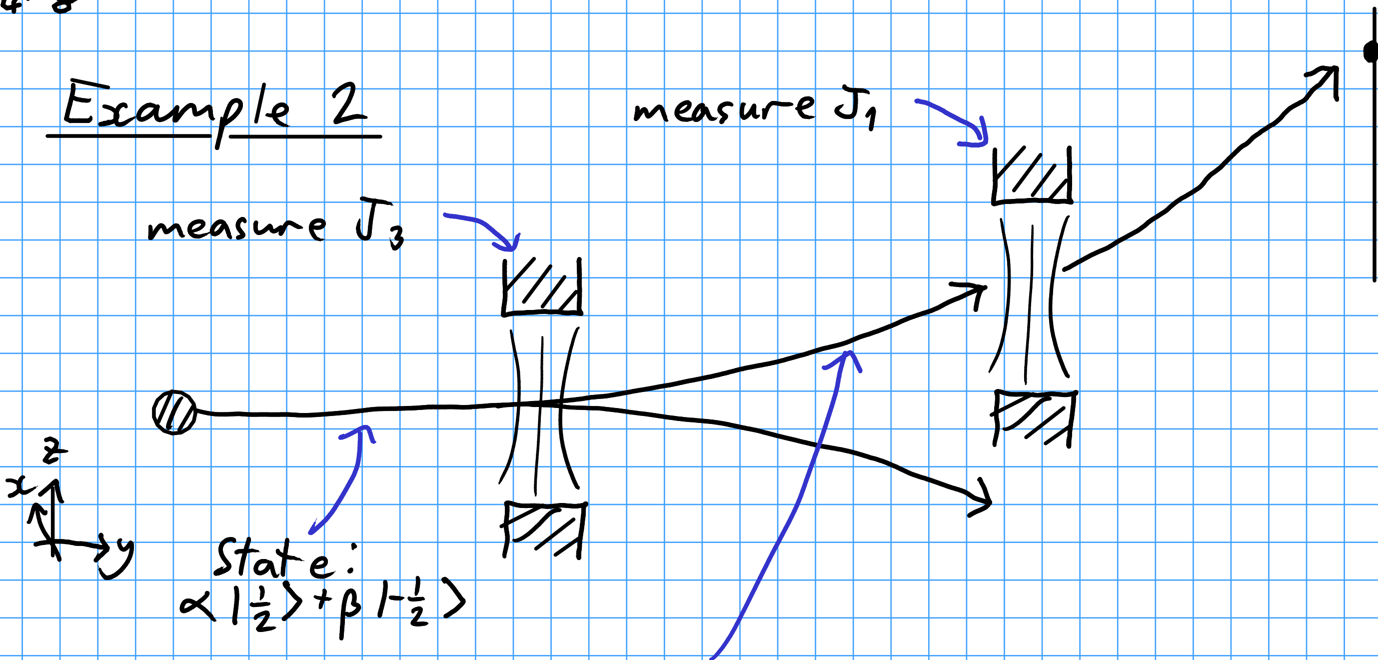
$$= |\alpha|^2$$

Similarly, prob. of outcome $-\frac{\hbar}{2}$ is $\langle\psi|(|-\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}|)|\psi\rangle = |\beta|^2$.

So, S-G apparatus splits beam into two, one corresponding to $+\frac{\hbar}{2}$ outcome of J_3 measurement, one to $-\frac{\hbar}{2}$ outcome.

Averaging over all orientations, i.e. over random α, β, by symmetry must get equal numbers of electrons in both beams.

Example 2



Post-measurement state after $+\frac{\hbar}{2}$ outcome of (first) J_3 measurement is corresponding eigvect. $|\tfrac{1}{2}\rangle$.

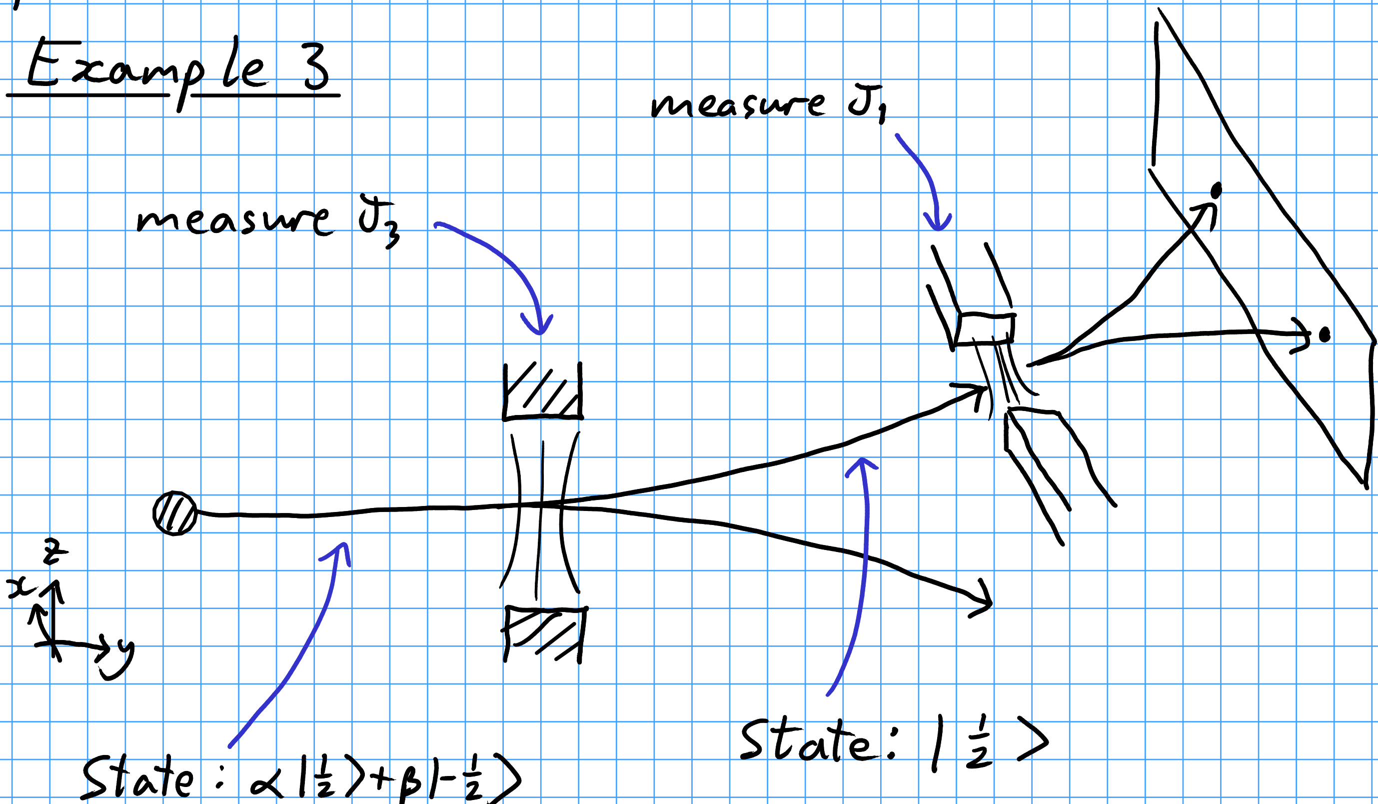
Second J_3 measurement outcome probabilites:

$$p(+\tfrac{\hbar}{2}) = |\langle \tfrac{1}{2}|\tfrac{1}{2}\rangle|^2 = \langle \tfrac{1}{2}|\left(|\tfrac{1}{2}\rangle\langle\tfrac{1}{2}|\right)|\tfrac{1}{2}\rangle = 1$$

$$p(-\tfrac{\hbar}{2}) = |\langle \tfrac{1}{2}|-\tfrac{1}{2}\rangle|^2 = \langle \tfrac{1}{2}|\left(|-\tfrac{1}{2}\rangle\langle-\tfrac{1}{2}|\right)|\tfrac{1}{2}\rangle = 0$$

i.e. <u>always</u> get same outcome $(+\frac{\hbar}{2})$ again.

Example 3



Second rotated S-G measures x-component J_1:

$$J_1 = \frac{\hbar}{2} X = \frac{\hbar}{2}\ |+\rangle\langle+| \ - \frac{\hbar}{2}\ |-\rangle\langle-|$$

$$|+\rangle = \frac{|\tfrac{1}{2}\rangle + |-\tfrac{1}{2}\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|\tfrac{1}{2}\rangle - |-\tfrac{1}{2}\rangle}{\sqrt{2}}$$

Probability of outcome $+\frac{\hbar}{2}$ is

$$p\left(\tfrac{\hbar}{2}\right) = \langle\tfrac{1}{2}|\left(|+\rangle\langle+|\right)|\tfrac{1}{2}\rangle = |\langle\tfrac{1}{2}|+\rangle|^2$$

Now

$$\langle\tfrac{1}{2}|+\rangle = \langle\tfrac{1}{2}|\left(\frac{|\tfrac{1}{2}\rangle + |-\tfrac{1}{2}\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

so

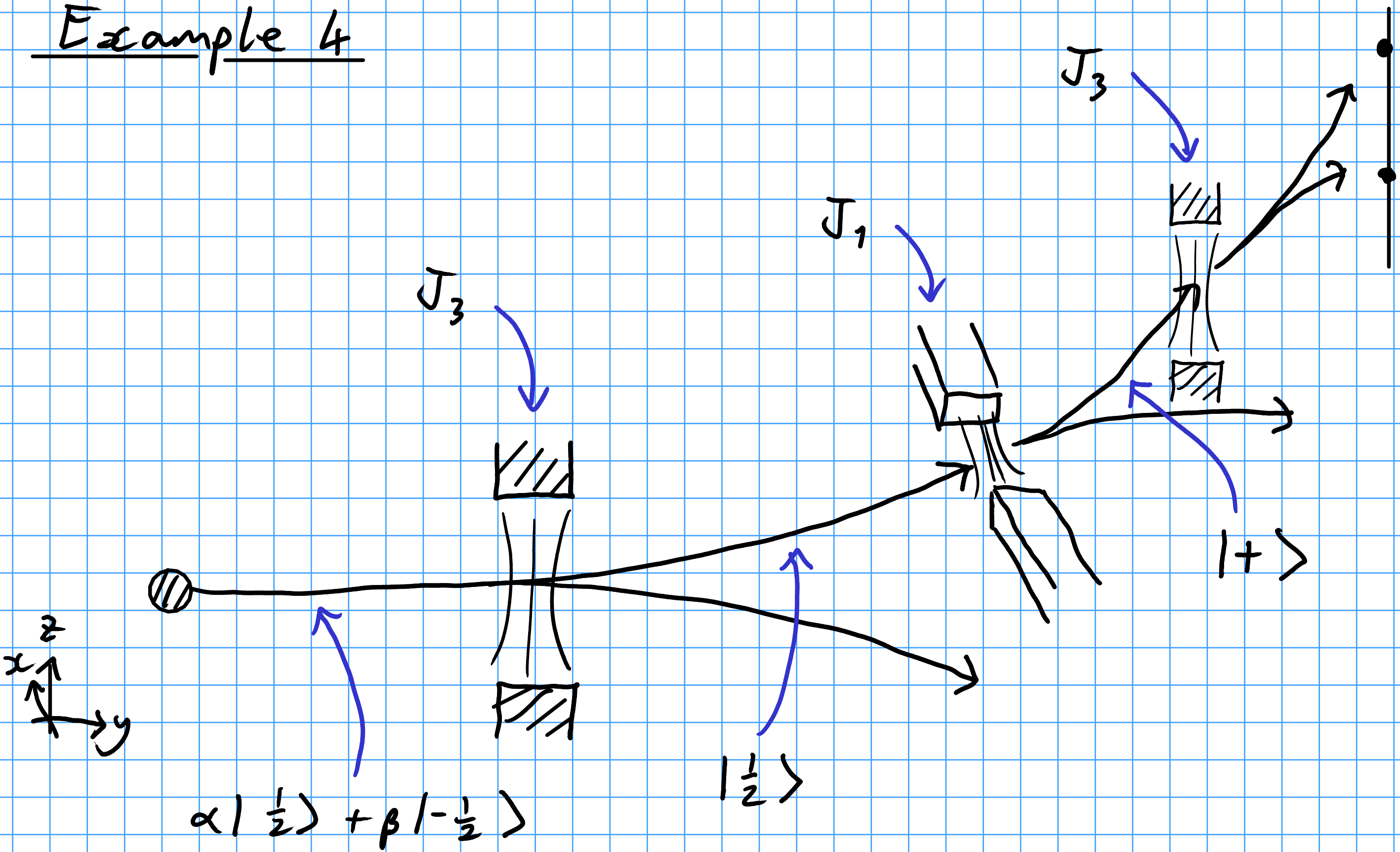
$$p\left(\tfrac{\hbar}{2}\right) = \frac{1}{2}$$

Similarly, probability of J_1 measurement giving outcome $-\frac{\hbar}{2}$ is $p(-\frac{\hbar}{2}) = \frac{1}{2}$.

Thus both outcomes of J_1 measurement occur with equal probability, and beam splits in two again.

Note that post-measurement state given $+\frac{\hbar}{2}$ outcome is $|+\rangle$, and given $-\frac{\hbar}{2}$ outcome is $|-\rangle$.

Example 4



Final J_3 measurement of $|+\rangle$ state:

$$p\left(\frac{\hbar}{2}\right) = \left|\langle \tfrac{1}{2} | + \rangle\right|^2 = \frac{1}{2}$$

$$p\left(-\frac{\hbar}{2}\right) = \left|\langle -\tfrac{1}{2} | + \rangle\right| = \frac{1}{2}$$

$\rightarrow$ beam splits again.

Notice how different this is to what we are used to from classical physics (the every day world).

When measure e.g. J_3 in $|+\rangle$ state, Q.M. says:

- even though know exactly what state is, cannot predict what will happen, just get probabilities of $\frac{1}{2}$, $\frac{1}{2}$ ~~for~~ the two possible outcomes;

- measurement process immediately "collapses" wavefunction into new state which depends on what we decide to measure.

↳ Intrinsic unpredictability of Nature

- Is this satisfactory?
  We expect a measurement to reveal a property of the system, and we don't expect it to completely alter the system

- Einstein was unhappy with this ("I cannot believe that God plays dice").

- Perhaps there are hidden degrees of freedom and if we knew them we could make exact (non-probabilistic) predictions...