III Orbital Angular Momentum

Everything we have shown about eigvals, eigvects, and representations has resulted from the angular momentum commutation relations, so applies to any general quantum angular momentum operators.

But we started off considering orbital angular mom:

$$L_i = \sum_{jk} \varepsilon_{ijk} X_j P_k$$

Do all representations derived for general ang. mom. operators exist for the case of orbital ang. mom. ?

Theorem 8

For orbital angular momentum, the values j and m must be integer.

<u>Proof</u> (standard version)

In spherical polar coordinates, position vector

$$\underline{x} = (r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r\cos\theta)$$

Using the chain rule

$$\frac{\partial}{\partial\phi} = \frac{\partial x_1}{\partial\phi}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial\phi}\frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial\phi}\frac{\partial}{\partial x_3}$$

$$= -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$$

Thus

$$-i\hbar\frac{\partial}{\partial\phi} = X_1 P_2 - P_2 X_1 = L_3$$

Now, $L_3 |\psi_m\rangle$ can be rewritten

$$\frac{\partial \psi_m}{\partial\phi} = im\,\psi_m \qquad\qquad |\psi_m\rangle \equiv \psi_m(r,\theta,\phi)$$

Integrating,

$$\psi(r,\theta,\phi) = e^{im\phi}\,\psi_m(r,\theta,0) \qquad [*]$$

On the other hand, we want the wavefunction ψ_m to be single-valued, so

$$\psi_m(r,\theta,\phi) = \psi_m(r,\theta,\phi+2\pi) = e^{2\pi i m}\,\psi_m(r,\theta,\phi)$$

Comparing with [*] either trivially $\psi = 0$, or m must be integer. $\square$

How can this be reconciled with e.g. the spin-$\frac{1}{2}$ representation we constructed?

Recall that global phases have no physical relevance in quantum mechanics, since $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ give identical expectation values for all observables:

$$\langle\psi|e^{-i\phi}\,\mathcal{O}\,e^{i\phi}|\psi\rangle = e^{-i\phi+i\phi}\langle\psi|\mathcal{O}|\psi\rangle$$
$$= \langle\psi|\mathcal{O}|\psi\rangle$$

So, abstractly, $|\psi_m\rangle$ and $-|\psi_m\rangle$ represent the same state. But wavefunctions are more restrictive, since they must assign a value to each point in space.

(More sophisticated explanation is that by insisting on wavefunctions, we are insisting on a particular (∞-dimensional!) Hilbert space for our angular momentum representations, and there's no reason to expect all representations to exist on that Hilbert space.)

So is the spin-$\frac{1}{2}$ representation a purely abstract, mathematical construction that has no physical realisation?

NO! Experiments (e.g. Stern-Gerlach) show that fundamental particles such as electrons have, in addition to any orbital angular momentum due to their motion, an <u>intrinsic</u> angular momentum, or "spin".

This intrinsic spin property will be our playground in which to demonstrate weird, amazing, and truely quantum effects.

Issues with Theorem 8 (non-examinable)

In proof, had to assume particular Hilbert space, and appeal to single-valuedness of wavefunction.

But why should we insist orbital ang. mom. lives in this Hilbert space? And why must the wavefunction be single-valued? (Really, only requirement is that <u>observables</u> be single-valued...)

Even such grandfathers of QM as Pauli and Bohm were concerned by this.

Here, we give an alternate, purely abstract, proof of Theorem 8 (due to Kaplan & Yu, 1971) that is both simple and in the spirit of our previous development of ang. mom.

Proof (sophisticated version)

Let

$$A = \frac{1}{\sqrt{2}}(P_1 - iX_1)$$

$$B = \frac{1}{\sqrt{2}}(P_2 - iX_2)$$

$$C = B + iA$$

Using the position–momentum commutation relations

$$[X_i, P_j] = i\hbar\,\delta_{ij}$$

$$[X_i, X_j] = [P_i, P_j] = 0$$

it is straightforward to verify that

(i) $[A, A^\dagger] = [B, B^\dagger] = \hbar$

(ii) $[C, C^\dagger] = 2\hbar$

(iii) $[A^\dagger A, B^\dagger B] = 0$

(iv) $[C^\dagger C, A^\dagger A + B^\dagger B] = 0$

(v) $C^\dagger C - (A^\dagger A + B^\dagger B) = L_3$.

Lemma 1

If $[A, A^\dagger] = \lambda$, the eigenvalues of $A^\dagger A$ are $0, \lambda, 2\lambda, \dots, n\lambda \quad n \in \mathbb{Z}_+$

Proof

$\left[\frac{A}{\sqrt{\lambda}}, \frac{A^\dagger}{\sqrt{\lambda}}\right] = 1$, so $A/\sqrt{\lambda}$, $A^\dagger/\sqrt{\lambda}$ obey the harmonic oscillator commutation relations.

Thus eigvals of $\frac{A^\dagger A}{\lambda}$ are $0, 1, 2 \dots n$ $\square$

Lemma 2

If $[A, B] = 0$, eigenvalues of $A \pm B$ are sums or differences of eigenvalues of A and B.

Proof

Since A & B commute, they have identical eigvects. Let $|\psi\rangle$ be any such eigvect. with eigvals. α, β resp.

Then

$$(A \pm B)|\psi\rangle = \alpha|\psi\rangle \pm \beta|\psi\rangle = (\alpha \pm \beta)|\psi\rangle$$

Returning to main proof,

(i) + Lemma 1 $\Rightarrow$ eigvals. $a\hbar$, $b\hbar$ of $A^\dagger A$, $B^\dagger B$ given by +ve integers $a, b \in \mathbb{Z}_+$

(ii) + Lemma 1 $\Rightarrow$ eigvals. $2c\hbar$ of $C^\dagger C$ are given by +ve, even integers, $2c \in \mathbb{Z}_+$

(iii), (iv) imply that we can apply Lemma 2 to (iv) to obtain for eigvals. of L_3

$$m\hbar = (2c - a - b)\hbar \qquad a, b, c \in \mathbb{Z}_+$$

Thus m must be integer $\square$