Quantum Mechanics
Lecture Notes, Part 1

Overview

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My part of course — beautiful / mysterious

- 1st half:
  angular mom. & spin — some of "most quantum" features
- 2nd half:
  use these to investigate deepest & most mysterious aspects of QM

Lectures

1 … 4, 5 [Angular momentum and spin

6 … 8 [Measurement; tensor products

9, 10 [Non-locality, entanglement & Bell inequalities

Bibliography

Any QM text book will cover this:

* Hannabus (Chapter 8)
Sakurai; Cohen-Tanudji; Messiah ...

For interest only (well beyond level of course):

Cornwell "Group Theory in Physics" (Chapter 12, Volume 2)

No single ideal text book:

Nielsen & Chuang (Chapter 2)

Wolf & Werner review paper
www.arXiv.org/abs/quant-ph/0107093

0. Introduction

Angular momentum

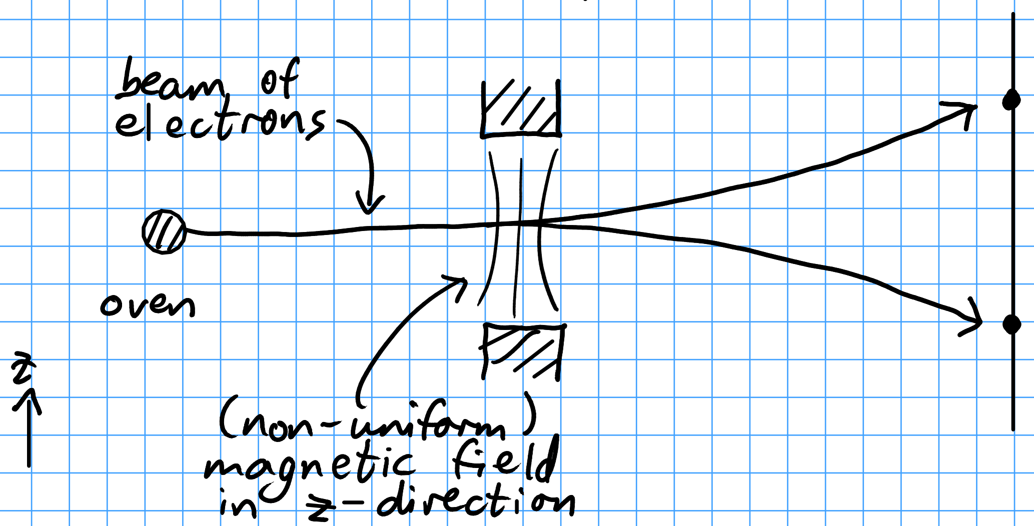
Very important in analysing many problems in classical mechanics, e.g. orbits of planets, collisions, central force problems...

Historically, important in one of key early successes of QM (before QM fully understood!):

Rutherford's model of atom as "mini solar system" + assuming angular momentum is quantized beautifully explains spectral emission & absorbtion patterns of (Hydrogen) atom [Bohr and others]

Spin

Stern Gerlach Experiment

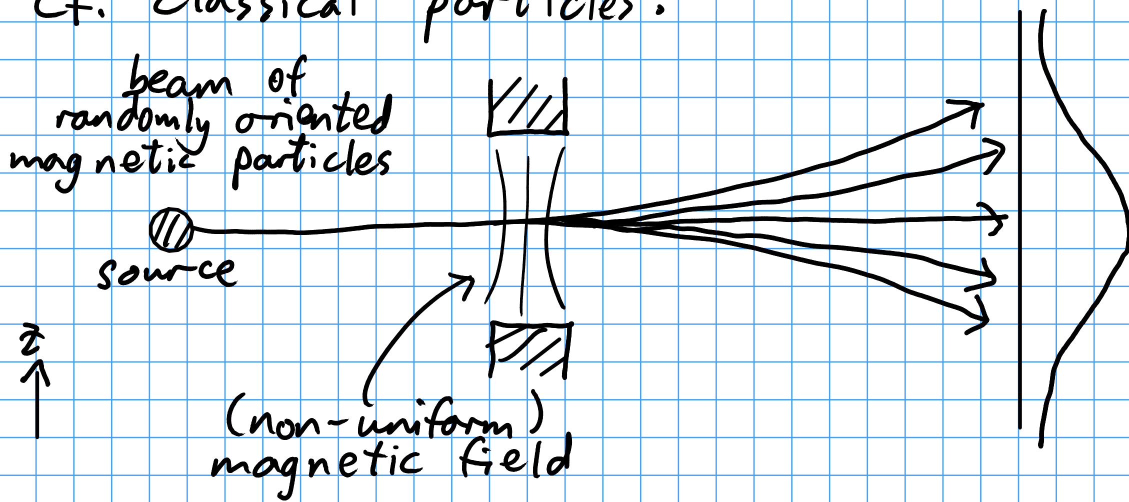


Electrons split into two beams, but **never** appear in between.

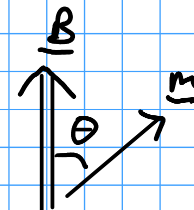
Electrons have intrinsic angular momentum, or "spin", resulting in an intrinsic magnetic moment (it's a charged particle).

S.G. apparatus measures z-component of spin. But why only two values?!?

Cf. classical particles:



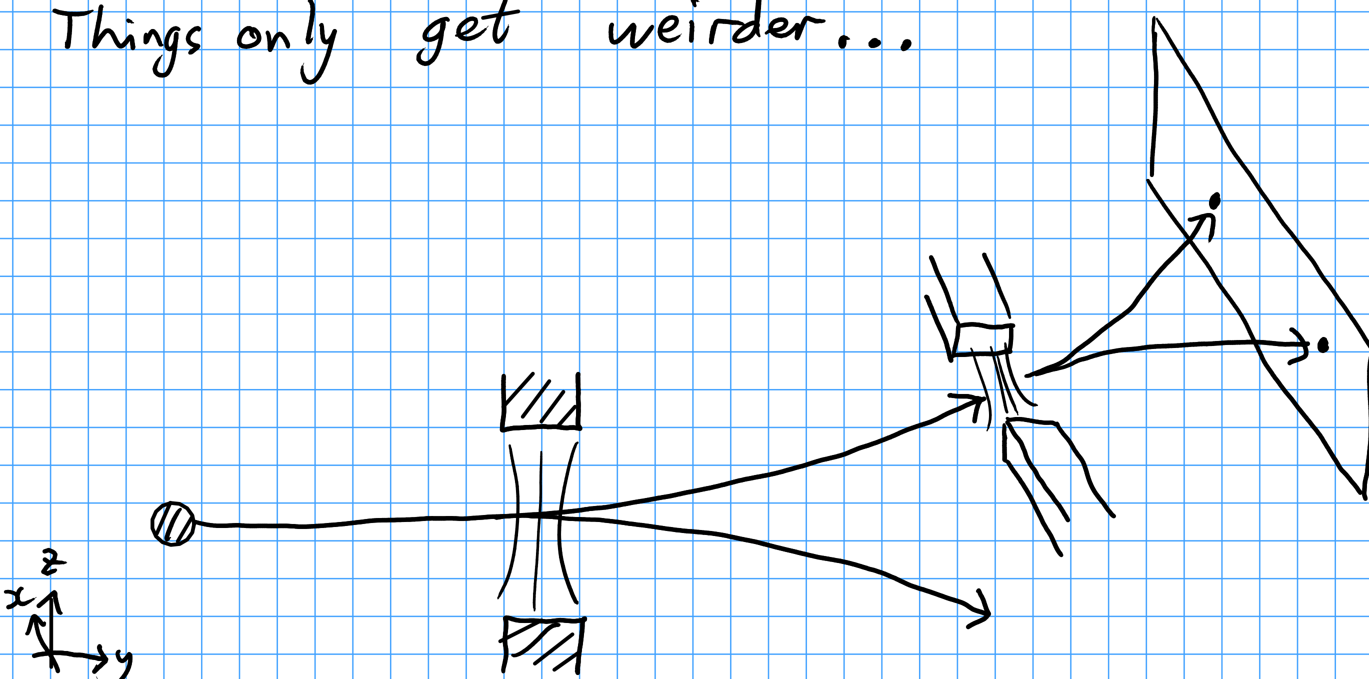
Deflection depends on component of magnetic moment in z-direction.



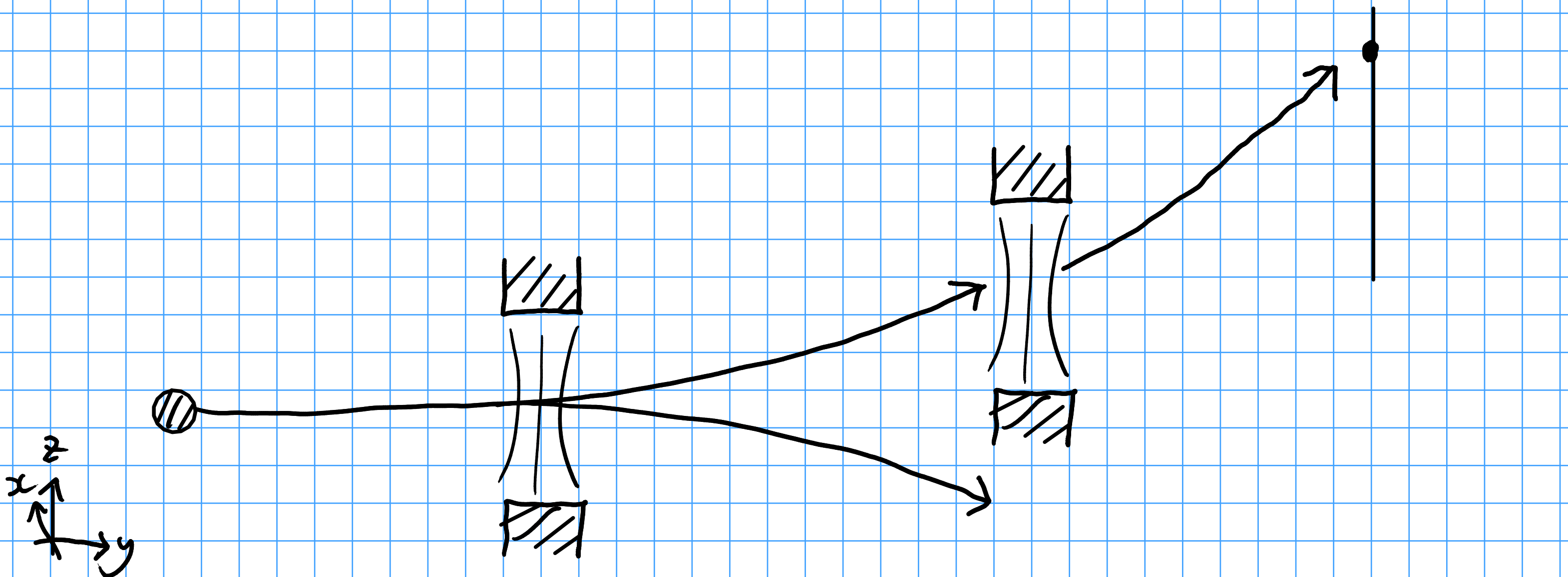
$\underline{m} \cdot \underline{B} = mB\cos\theta$

Get distribution of particles due to randomly distributed orientation θ

Things only get weirder...

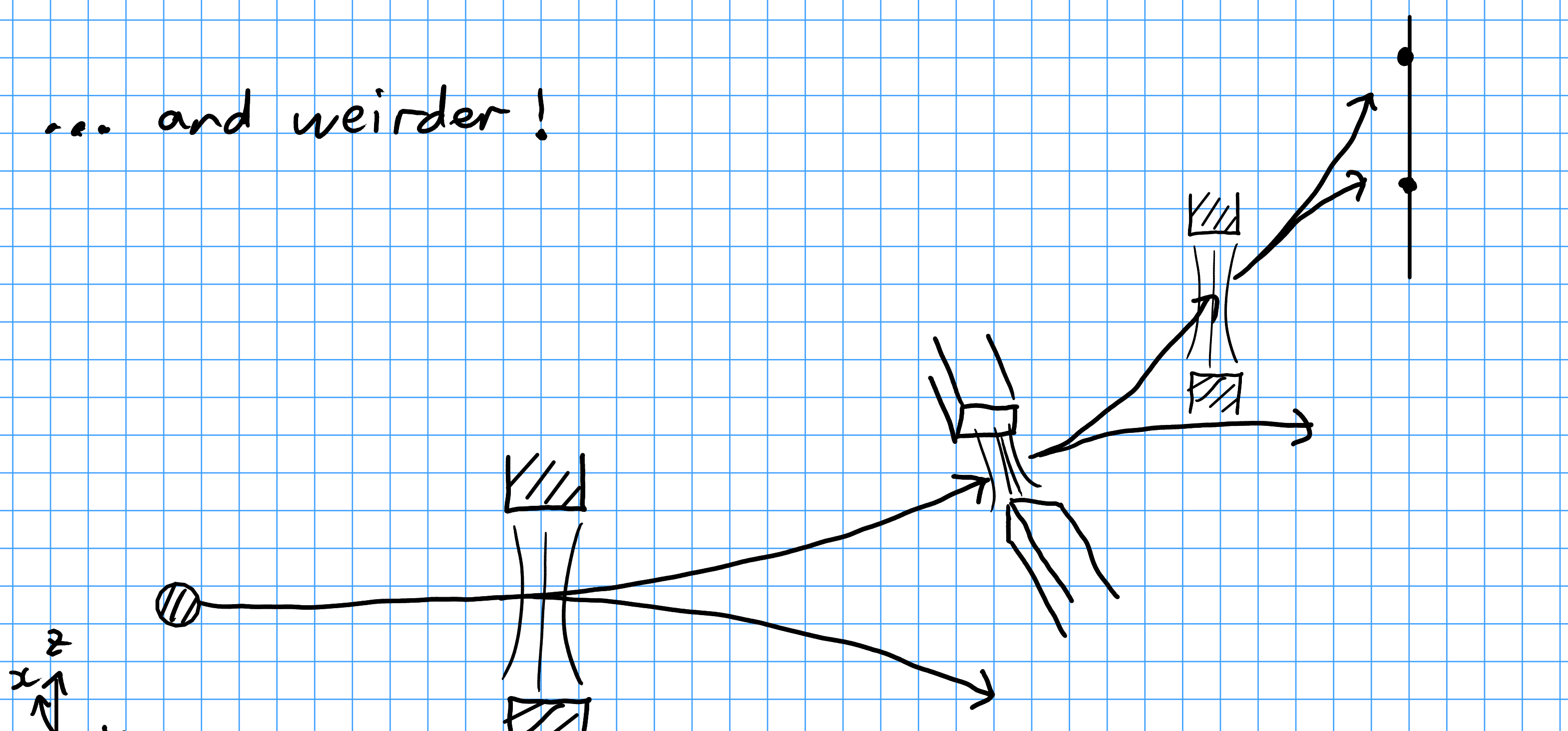


If take upper beam and feed it through S. G. apparatus oriented along x-axis, beam splits into two (and only two) again!



If we pass one branch through second z-oriented SG, get only one beam as expected.

... and weirder!



But if we put an x-oriented SG in between, second z-oriented SG goes back to splitting beam in two again, as if first one wasn't there!

Clearly, spin doesn't behave like anything we know from classical physics!

I. Angular Momentum

Classical

Recall in 3 dimensions it is a vector

$$\underline{L} = \underline{x} \wedge \underline{p} \qquad \underline{x} \text{ position}, \quad \underline{p} \text{ momentum}$$

or in Cartesian coordinates, x, y, z components are

$$L_1 = x_2 p_3 - x_3 p_2 \qquad (x \text{ component})$$
$$L_2 = x_3 p_1 - x_1 p_3 \qquad (y \quad " \quad)$$
$$L_3 = x_1 p_2 - x_2 p_1 \qquad (z \quad " \quad)$$

Can write this more concisely using alternating symbol ε_{ijk} (also called "Levi-Civita symbol" or "anti-symmetric tensor"):

Def$^{\underline{n}}$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is cyclic perm}^{\underline{n}} \text{ of } (123) \\ -1 & \;"\; (ijk) \;"\; \;"\; \;"\; (321) \\ 0 & \text{otherwise} \end{cases}$$

$i, j, k = 1, 2, 3$

Then

$$L_i = \sum_{jk} \varepsilon_{ijk} x_j p_k$$

Quantum

Position & momentum become operators X & P is QM

Suggests ang. mom. should be described by 3-component vector of operators.

Def^{n}

Quantum mechanical orbital angular momentum vector $\underline{L}$ has components

$$L_1 = X_2 P_3 - X_3 P_2$$
$$L_2 = X_3 P_1 - X_1 P_3$$
$$L_3 = X_1 P_2 - X_2 P_1$$

or more concisely

$$L_i = \sum_{jk} \varepsilon_{ijk} X_j P_k$$

Proposition 1

(i) $[L_j, P_k] = i\hbar \sum_l \varepsilon_{jkl} P_l$

(ii) $[L_j, X_k] = i\hbar \sum_l \varepsilon_{jkl} X_l$

(iii) $[L_j, L_k] = i\hbar \sum_l \varepsilon_{jkl} L_l$

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Lemma 2

For any operators A, B, C:

$$[AB, C] = A[B, C] + [A, C]B$$

$$[C, AB] = A[C, B] + [C, A]B$$

Proof

$$\begin{aligned}[AB, C] &= ABC - CAB \\ &= A([B, C] + CB) - CAB \\ &= A[B, C] + ([A, C] + CA)B - CAB \\ &= A[B, C] + [A, C]B\end{aligned}$$

$[C, AB]$ (excercise) $\square$

Proof of Prop$^{\underline{n}}$ 1

(i) $$\begin{aligned}[L_j, P_k] &= [\sum \varepsilon_{jmn} X_m P_n , P_k] \\ &= \sum \varepsilon_{jmn} [X_m P_n, P_k] \\ &= \sum \varepsilon_{jmn} (X_m [P_n, P_k] + [X_m, P_k] P_n) \quad \text{by Lemma 2} \\ &= \sum \varepsilon_{jmn} (0 + i\hbar \delta_{mk} P_n) \\ &= i\hbar \sum \varepsilon_{jkn} P_n \end{aligned}$$

(ii) Similar (exercise)

(iii) $$\begin{aligned}[L_1, L_2] &= [L_1, X_3 P_1 - X_1 P_3] \\ &= X_3 [L_1, P_1] + [L_1, X_3] P_1 \\ &\quad - (X_1 [L_1, P_3] - [L_1, X_1] P_3) \quad \text{using Lemma 2} \\ &= 0 - i\hbar X_2 P_1 + i\hbar X_1 P_2 - 0 \quad \text{using parts (i) \& (ii)} \\ &= i\hbar L_3 \end{aligned}$$

Relations for other components follow by permuting indices. $\square$

The <u>commutation relations</u> in (iii) are particularly important. We will see that essentially <u>all</u> properties of ang. mom. arise as consequences of these.

Motivates following abstraction:

Def$^{\underline{n}}$

Any three operators J_1, J_2, J_3 that satisfy

$$[J_j, J_k] = i\hbar \sum_l \varepsilon_{jkl} J_l$$

are called (quantum) angular momentum operators.

J_1, J_2, J_3 are components of quantum ang. mom. in Cartesian coordinates.

But $[J_1, J_2] \neq 0$ etc. means they are not simultaneously measurable in general.
(I.e. expect to find states for which J_1, J_2, J_3 do not simultaneously have well-defined values, cf. position & momentum.)

How can ang. mom. be constant of motion if components are not simultaneously measurable? But it is <u>total</u> ang. mom. which we expect to be conserved in central force problems...

Def$^{\underline{n}}$

Total angular momentum is defined by

$$J^2 = J_1^2 + J_2^2 + J_3^2$$

Proposition 3

$$[J^2, J_i] = 0$$

Proof

$$[J^2, J_i] = [\sum_j J_j J_j, J_i]$$

$$= \sum_j J_j [J_j, J_i] + \sum_j [J_j, J_i] J_j$$

Lemma 2 again

$$= i\hbar \sum_{jk} J_j \varepsilon_{jik} J_k + i\hbar \sum_{jk} \varepsilon_{jik} J_k J_j$$

$$= i\hbar \sum_{jk} \underset{\text{antisymmetric in } j,k}{\varepsilon_{jik}} \underbrace{(J_j J_k + J_k J_j)}_{\text{symmetric in } j,k}$$

$$= 0 \qquad \square$$

(Explicitly:

$$\sum_{jk} \varepsilon_{jik} J_k J_j = \sum_{kj} \varepsilon_{kij} J_j J_k$$

Just relabeling $k \leftrightarrow j$
expression has not changed.

$$= -\sum_{kj} \varepsilon_{jik} J_j J_k$$

antisymmetric, so swapping two indices picks up minus sign

So $\sum_{jk} (\varepsilon_{jik} J_j J_k + \varepsilon_{jik} J_k J_j)$

$$= \sum_{jk} (\varepsilon_{jik} J_j J_k - \varepsilon_{jik} J_j J_k) = 0 \quad)$$

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Since $[J^2, J_i] = 0$, we can find simultaneous eigenstates of J^2 and e.g. J_3, in which both have well-defined values.