II. Combining quantum systems: Multiple Particles & Tensor Products

Aim to describe space of states of two (or more) systems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, and also to describe operators acting on this space.

Amongst these states, expect to find states in which system A is in state $|\psi_A\rangle$, and system B in state $|\psi_B\rangle$.

However, as we will soon see, linearity of Q.M. implies not all states of combined system are of this form
→ entanglement, perhaps the strangest & most remarkable feature of Q.M.

Tensor product state spaces

Imagine two systems A and B that are spatially separated

A $\mathcal{H}_A$ basis $\{|a_i\rangle\}$

B $\mathcal{H}_B$ basis $\{|b_i\rangle\}$

How do we describe states of joint system AB?

Natural starting point is $\mathcal{H}_A \times \mathcal{H}_B$.
But we need a <u>Hilbert space</u> (Postulate 1).

Get vector space* by taking space spanned by all pairs of basis states $(|a_i\rangle, |b_j\rangle) \in \mathcal{H}_A \times \mathcal{H}_B$.
("Free vector space over $(|a_i\rangle, |b_j\rangle)$" if we want to sound impressive!)

*We'll worry about the inner product later...

Introduce <u>tensor product</u> of $\mathcal{H}_A$ and $\mathcal{H}_B$, denoted "$\mathcal{H}_A \otimes \mathcal{H}_B$", to describe this joint vector space.

Denote $(|a_i\rangle, |b_j\rangle)$ by "$|a_i\rangle \otimes |b_j\rangle$".
Arbitrary vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ is then

$$\sum_{ij} \alpha_{ij} \, |a_i\rangle \otimes |b_j\rangle$$

<u>Example:</u> $\mathbb{C}^2 \otimes \mathbb{C}^2$

basis $\{|0\rangle, |1\rangle\}$ basis $\{|0\rangle, |1\rangle\}$

$\hookrightarrow$ 4 basis elements:

$$|0\rangle \otimes |0\rangle \xrightarrow{\text{shorthand}} |0\rangle|0\rangle \text{ or just } |00\rangle$$
$$|0\rangle \otimes |1\rangle \qquad |0\rangle|1\rangle \qquad |01\rangle$$
$$|1\rangle \otimes |0\rangle \qquad |1\rangle|0\rangle \qquad |10\rangle$$
$$|1\rangle \otimes |1\rangle \qquad |1\rangle|1\rangle \qquad |11\rangle$$

Arbitrary vector is

$$\sum_{i,j=0}^{1} \alpha_{ij} |i\rangle|j\rangle$$

$$= \alpha_{00} |0\rangle|0\rangle + \alpha_{01} |0\rangle|1\rangle + \alpha_{10} |1\rangle|0\rangle + \alpha_{11} |1\rangle|1\rangle$$

$\to$ $\mathbb{C}^2 \otimes \mathbb{C}^2$ is (complex) vector space with 4 basis elements, thus $\mathbb{C}^2 \otimes \mathbb{C}^2 \hat{=} \mathbb{C}^4$.

This construction seems reasonable, but is it well defined? Is there some other way of constructing a joint Hilbert space from $\mathcal{H}_A$, $\mathcal{H}_B$? Does it depend on our choice of bases?

<u>Theorem 1</u> (Universal property of tensor product)

$\mathcal{H}_A \otimes \mathcal{H}_B$ is the <u>unique</u> vector space (up to isomorphism) satisfying:

(i) $\exists$ bilinear map $i : \mathcal{H}_A \times \mathcal{H}_B \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_B$

(ii) $\forall$ bilinear maps $f : \mathcal{H}_A \times \mathcal{H}_B \longrightarrow \mathcal{H}_C$,
$\exists$ unique linear map (operator) $L_f : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_C$ s.t. $L_f \circ i = f$.

Proof

Let $i_1 : \mathcal{H}_A \times \mathcal{H}_B \longrightarrow (\mathcal{H}_A \otimes \mathcal{H}_B)_1$

$\quad\; i_2 : \mathcal{H}_A \times \mathcal{H}_B \longrightarrow (\mathcal{H}_A \otimes \mathcal{H}_B)_2$

define two tensor product spaces satisfying (i) & (ii).

Set $\mathcal{H}_C = (\mathcal{H}_A \otimes \mathcal{H}_B)_2$, and take

$f = i_2 : \mathcal{H}_A \times \mathcal{H}_B \to (\mathcal{H}_A \otimes \mathcal{H}_B)_2$

By assumption, have linear map

$L_{i_2} : (\mathcal{H}_A \otimes \mathcal{H}_B)_1 \to (\mathcal{H}_A \otimes \mathcal{H}_B)_2$

s.t. $L_{i_2} \circ i_1 = f = i_2$.

Swapping roles of $(\mathcal{H}_A \otimes \mathcal{H}_B)_1$ & $(\mathcal{H}_A \otimes \mathcal{H}_B)_2$ similarly gives

$L_{i_1} : (\mathcal{H}_A \otimes \mathcal{H}_B)_2 \longrightarrow (\mathcal{H}_A \otimes \mathcal{H}_B)_1$

s.t. $L_{i_1} \circ i_2 = i_1$.

I.e. the following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}_A \times \mathcal{H}_B \arrow[r, "i_1"] \arrow[d, "i_2"'] & (\mathcal{H}_A \otimes \mathcal{H}_B)_1 \arrow[dl, shift right, "L_{i_2}"'] \\
(\mathcal{H}_A \otimes \mathcal{H}_B)_2 \arrow[ur, shift right, "L_{i_1}"'] &
\end{tikzcd}$$

Thus $L_2 = L_{i_2} \circ L_{i_1} : (\mathcal{H}_A \otimes \mathcal{H}_B)_2 \longrightarrow (\mathcal{H}_A \otimes \mathcal{H}_B)_2$ is linear map s.t. $L_2 \circ i_2 = i_2$.

But, applying (ii) to $i_2 : \mathcal{H} \times \mathcal{H} \longrightarrow (\mathcal{H}_A \otimes \mathcal{H}_B)_2$, we see that L_2 is the <u>unique</u> map s.t. $L_2 \circ i_2 = i_2$

Clearly the identity map fulfils this property, so by uniqueness $L_{i_2} \circ L_{i_1} = L_2 = \mathrm{id}$.

Similarly, $L_{i_1} \circ L_{i_2} = L_1 = \mathrm{id}$.

Thus L_{i_1} & L_{i_2} are inverse isomorphisms, and $(\mathcal{H}_A \otimes \mathcal{H}_B)_1 \cong (\mathcal{H}_A \otimes \mathcal{H}_B)_2$.

Remains to show our construction of $\mathcal{H}_A \otimes \mathcal{H}_B$ as span $\{(|a_i\rangle, |b_j\rangle)\}$ fulfils (i) & (ii).

Let f be a bilinear map on $\mathcal{H}_A \times \mathcal{H}_B$,

$$|a\rangle = \sum_i \alpha_i |a_i\rangle \in \mathcal{H}_A,$$
$$|b\rangle = \sum_j \beta_j |b_j\rangle \in \mathcal{H}_B.$$

$$
\begin{aligned}
f(|a\rangle, |b\rangle) &= \sum_i \alpha_i f(|a_i\rangle, |b\rangle) && \text{by linearity in 1st argument} \\
&= \sum_i \alpha_i \sum_j \beta_j f(|a_i\rangle, |b_j\rangle) && \text{'' '' '' 2nd ''} \\
&= \sum_{ij} \alpha_i \beta_j f(|a_i\rangle, |b_j\rangle)
\end{aligned}
$$

thus speeifying the values on $\{(|a_i\rangle, |b_j\rangle)\}$ defines f.

Conversely, since $\{(|a_i\rangle, |b_j\rangle)\}$ forms a basis, any set of values $f(|a_i\rangle, |b_j\rangle)$ gives a bilinear function.

Therefore bilinear functions are in 1-to-1 correspondance with linear maps on $\mathcal{H}_A \otimes \mathcal{H}_B$, fulfilling (ii).

Need bilinear map $i: \mathcal{H}_A \times \mathcal{H}_B \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_B$.
Trivial using our 1-to-1 correspondance (e.g. identity map will do the job!) $\square$

In fact, usually we take Theorem 1 to be the <u>definition</u> of the tensor product.

Separate evolution (or measurement) of the individual systems A & B is, according to Postulates 2 (or 3), described by linear operators on $\mathcal{H}_A$ and $\mathcal{H}_B$, which together give a bilinear map on $\mathcal{H}_A \times \mathcal{H}_B$.

Thus Theorem 1 shows that $\mathcal{H}_A \otimes \mathcal{H}_B$ is the unique way (up to isomorphism) of combining $\mathcal{H}_A$ & $\mathcal{H}_B$ to produce a Hilbert space for the joint system, such that evolution and measurement are described by a <u>linear</u> operator on the joint Hilbert space (as required by Postulates 2 & 3 applied to the joint system).

Of course, nothing (not even Thm 1!) forces Nature to actually behave this way.

Experiments <u>tell</u> us that quantum mechanics appears to be linear, hence state spaces of multiple quantum systems combine as tensor products.

Tensor product is <u>bilinear</u> by definition:

If

$$|\psi_A\rangle = a_0|0\rangle + a_1|1\rangle$$
$$|\psi_B\rangle = b_0|0\rangle + b_1|1\rangle$$

then state of combined system is

$$|\psi_A\rangle \otimes |\psi_B\rangle = (a_0|0\rangle + a_1|1\rangle)\otimes(b_0|0\rangle + b_1|1\rangle)$$
$$= a_0b_0|0\rangle|0\rangle + a_0b_1|0\rangle|1\rangle + a_1b_0|1\rangle|0\rangle + a_1b_1|1\rangle|1\rangle$$

[Or, in vector notation,

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \text{“}\begin{pmatrix} a_0 \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \\ a_1 \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \end{pmatrix}\text{”} = \begin{pmatrix} a_0b_0 \\ a_0b_1 \\ a_1b_0 \\ a_1b_1 \end{pmatrix}$$
]

<u>Definition</u> (product state)

A state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a <u>product state</u> if it can be written $|\psi\rangle = |\varphi_A\rangle \otimes |\phi_B\rangle$.

<u>However</u>, it is fundamentally important that not all states in $\mathcal{H}_A \otimes \mathcal{H}_B$ are product states

E.g.

Claim

$$|\phi\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$ is not a product state

Proof

If $|\phi\rangle$ was product state, could be written

$(a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$

$= a_0 b_0 |0\rangle|0\rangle + a_0 b_1 |0\rangle|1\rangle + a_1 b_0 |1\rangle|0\rangle + a_1 b_1 |1\rangle|1\rangle \quad (*)$

for some a_0, a_1, b_0, b_1.

Now $|0\rangle|1\rangle$ term in $(*)$ implies $a_0 b_1 = 0$,

i.e. $a_0 = 0$ or $b_1 = 0$.

But then $|0\rangle|0\rangle$ or $|1\rangle|1\rangle$ term would vanish!

$\longrightarrow$ contradiction.

Definition 1

Any state which cannot be written in product form is said to be <u>entangled</u>.

Careful!

$$\frac{|0\rangle|0\rangle + |1\rangle|1\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle}{2}$$

$$= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

so it <u>is</u> of product form.

It is a deep fact about the way Nature appears to behave that spaces of many particles combine via a tensor product, implying the existence of entangled states.

Inner Products

Still need an inner product on $\mathcal{H}_A \otimes \mathcal{H}_B$ to make it into a Hilbert space, as required by Postulate 1 !

Definition 2

Inner product on $\mathcal{H}_A \otimes \mathcal{H}_B$ defined in terms of inner products $\langle v_A | w_A \rangle$ on $\mathcal{H}_A$ and $\langle v_B | w_B \rangle$ on $\mathcal{H}_B$ by

$$(\langle v_A | \langle v_B |)\,(|w_A\rangle |w_B\rangle) = \langle v_A | w_A \rangle \langle v_B | w_B \rangle$$

Def. 2 is sufficient to calculate all inner products, via linearity, e.g.

$$(\langle v_A | \langle v_B |)\,(\alpha |w_A\rangle |w_B\rangle + \beta |z_A\rangle |z_B\rangle)$$

$$= \alpha \langle v_A | w_A \rangle \langle v_B | w_B \rangle + \beta \langle v_A | z_A \rangle \langle v_A | z_B \rangle$$

Exercise: Verify that Def. 2 satisfies the defining properties of an inner product. (Easy!)

Example

$$(\langle 0|\langle 1|)(|0\rangle|0\rangle) = \langle 0|0\rangle\langle 1|0\rangle = 1 \cdot 0 = 0$$

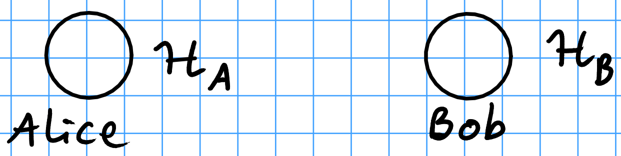
Example

Inner product of $|0\rangle|1\rangle + i|1\rangle|0\rangle$
with $|0\rangle|1\rangle + i|1\rangle|0\rangle$ is

$$(\langle 0|\langle 1| - i\langle 1|\langle 0|)(|0\rangle|1\rangle + i|1\rangle|0\rangle) = 0$$

note sign change
from Hermitian conjugate

Operators on Tensor Product Spaces



Operator A acts on $\mathcal{H}_A$

" B " " $\mathcal{H}_B$

Define linear operator $A \otimes B$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$:

- on basis $|a_i\rangle \otimes |b_j\rangle$

$$(A \otimes B)\ |a_i\rangle |b_j\rangle = A|a_i\rangle \otimes B|b_j\rangle$$

- on general state, extend by linearity:

$$(A \otimes B) \sum_{ij} \alpha_{ij}\ |a_i\rangle\ |b_j\rangle = \sum_{ij} \alpha_{ij}\ A|a_i\rangle \otimes B|b_j\rangle$$

Physically, this corresponds to Alice applying operator A to her system & Bob applying operator B to his system.

Example

Pauli operators on $\mathbb{C}^2$

$$X|0\rangle = |1\rangle \qquad Y|0\rangle = i|0\rangle \qquad Z|0\rangle = |0\rangle$$
$$X|1\rangle = |0\rangle \qquad Y|1\rangle = -i|1\rangle \qquad Z|1\rangle = -|1\rangle$$

Can define $X \otimes Z$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$, e.g.

$$X \otimes Z\ |0\rangle|0\rangle = X|0\rangle \otimes Z|0\rangle = |1\rangle|0\rangle$$

$$X \otimes Z\ |1\rangle|1\rangle = -|0\rangle|1\rangle$$

etc.

Thus

$$X \otimes Z \left(\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} \right) = \frac{|1\rangle|0\rangle - |0\rangle|1\rangle}{\sqrt{2}}$$

Note:

$X \otimes Z \neq Z \otimes X$, e.g.

$$Z \otimes X |0\rangle|0\rangle = |0\rangle|1\rangle \neq |1\rangle|0\rangle = X \otimes Z \; |0\rangle|0\rangle$$

orthogonal!

<u>Example</u>

Operator corresponding to doing nothing is identity operator $\mathbb{1}$:

$$\mathbb{1} \, |\psi\rangle = |\psi\rangle \qquad \forall \; |\psi\rangle$$

Thus operator corresponding to Alice doing X and Bob doing nothing is $X \otimes \mathbb{1}$:

$$X \otimes \mathbb{1} \; |0\rangle|0\rangle = |1\rangle|0\rangle$$

$$X \otimes \mathbb{1} \left(\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} \right) = \frac{|1\rangle|0\rangle + |0\rangle|1\rangle}{\sqrt{2}}$$

etc.

Proposition 3

If operator A has eigvals. λ_i and eigvects $|a_i\rangle$
B " " μ_i " " $|b_i\rangle$

then $A \otimes B$ has eigvals $\lambda_i \mu_j$ and eigvects. $|a_i\rangle|b_j\rangle$

Proof

$$A \otimes B \; |a_i\rangle|b_j\rangle = A|a_i\rangle \otimes B|b_j\rangle$$
$$= \lambda_i |a_i\rangle \otimes \mu_j |b_j\rangle$$
$$= \lambda_i \mu_j \; |a_i\rangle|b_j\rangle$$

Note that this implies $A \otimes B$ has spectral decomposition

$$A \otimes B = \sum_{ij} \lambda_i \mu_j \; |a_i\rangle|b_j\rangle\langle a_i|\langle b_j|$$
$$= \left(\sum_i \lambda_i \, |a_i\rangle\langle a_i|\right) \otimes \left(\sum_j \mu_j \, |b_j\rangle\langle b_j|\right)$$

as expected

Alternatively, as matrices we have e.g.

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \qquad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

$$A \otimes B = \text{``}\begin{pmatrix} a_{00}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{10} \end{pmatrix} & a_{01}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \\ a_{10}\begin{pmatrix} & '' & \end{pmatrix} & a_{11}\begin{pmatrix} & '' & \end{pmatrix} \end{pmatrix}\text{''}$$

$$= \begin{pmatrix} a_{00}b_{00} & a_{00}b_{01} & a_{01}b_{00} & a_{01}b_{01} \\ a_{00}b_{10} & a_{00}b_{11} & & \\ a_{10}b_{00} & a_{10}b_{01} & \text{etc.} & \\ a_{10}b_{10} & a_{10}b_{11} & & \end{pmatrix}$$

Proposition 4

$$(A \otimes B)^{+} = A^{+} \otimes B^{+}$$

Proof: Easy exercise.
(Show this on a basis — extends by linearity)

Proposition 5

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

Proof: Another easy exercise
(Again, show on basis, extend by linearity).

Note: have discussed tensor products of 2 spaces, but everything extends in obvious way to 3 and more spaces.