Hamiltonian Simulation

So far, we have required the entire physics of H to be exactly replicated in the relevant subspace of $H' = \mathcal{E}(H)$.

It is time to relax this, and allow the physics to be replicated approximately up to some small error.

Also, so far we have required the image to have 0 support outside of the encoding subspace. Often we do not care what happens outside of the correct subspace. So we should also relax this.

Typically, we want the encoding subspace to be the "low-energy" subspace of H', i.e. the subspace of states below some energy cutoff Δ (which however we typically want to be large, so "low energy" is only relative to this high-energy cutoff).

Def. Local Simulation

Let $H' \in \mathrm{Herm}\left(\bigotimes_{i=1}^{n} \mathcal{H}_i'\right)$,

$H \in \mathrm{Herm}\left(\bigotimes_{i=1}^{n} \mathcal{H}_i\right)$

We say H' $(\Delta, \varepsilon, \eta)$-approximately simulates H if $\exists$ local encoding

$$\mathcal{E}(H) = V(H \otimes P + \bar{H} \otimes Q)V^\dagger$$

s.t.

(i) $\exists$ encoding $\tilde{\mathcal{E}}(H) = \tilde{V}(H \otimes P + \bar{H} \otimes Q)\tilde{V}^\dagger$ (not necessarily local) s.t

- $\tilde{\mathcal{E}}(\mathbb{1}) =$ projector onto eigenspace of H' with eigvals $\leq \Delta$ $=: S_{\leq\Delta}$
- $\| V - \tilde{V} \| \leq \eta$

(ii) $\left\| H'|_{\leq\Delta} - \tilde{\mathcal{E}}(H) \right\| \leq \varepsilon$

where $H'|_{\leq\Delta}$ denotes the restriction of H' to the subspace $S_{\leq\Delta}$

Think of $\mathcal{E}$ as the (local) encoding you know how to construct, but which doesn't exactly map into the low-energy subspace.

Meanwhile, $\tilde{\mathcal{E}}$ is an encoding which maps exactly into the low-energy subspace, but which you don't know much about the structure.

Since $\mathcal{E}$ & $\tilde{\mathcal{E}}$ are close to one another, they can be used interchangably, at the cost of making small approximation errors.

The trick to putting this to good use is to bounce between $\mathcal{E}$ and $\tilde{\mathcal{E}}$ depending on which set of properties one needs.

Example (Complex-to-real simulation)

H k-local qubit Hamiltonian

w.l.o.g. each term $h = \bigotimes_{j=1}^{k} \sigma_j$ where $\sigma_j \in \{X, Y, Z\}$

Let $\varphi_i(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$

$$\varphi_i(\sigma_{x,z}) = \sigma_{x,z} \oplus \sigma_{x,z} = \mathbb{1} \otimes \sigma_{x,z}$$

$$\varphi_i(\sigma_y) = J(\sigma_y \oplus \sigma_y) = \sigma_y \otimes \sigma_y$$

Construct

$$\mathcal{E}(h) = \bigotimes_i \varphi_i(\sigma_i)$$ manifestly local

$$= \bigotimes_{j=1}^{k} \big(|+_y\rangle\langle +_y|_j \otimes \sigma_j + |-_y\rangle\langle -_y| \otimes \bar{\sigma}_j \big).$$

Let $S := \mathrm{span}\{ |+_y\rangle^{\otimes n}, |-_y\rangle^{\otimes n} \}$

$$\tilde{H} = \sum_i \mathcal{E}(h_i)$$

$$\tilde{H}|_S = |+_y\rangle\langle +_y|^{\otimes n} \otimes H + |-_y\rangle\langle -_y| \otimes \bar{H}$$

Let $H_0 := \sum_i (Y_i Y_{i+1} + \mathbb{1})$

H_0 is 0 on S and $\geq \mathbb{1}$ on $S^\perp$

$\Rightarrow H' = \tilde{H} + \Delta H_0$ is real and is a local $(\Delta, 0, 0)$-simulation of H.

"perfect simulation"

We now collect without proof some important properties of simulation (see [BH'14] & [CMP'19] for proofs].

Thm (Composition of simulations)
If A $(\Delta_A, \varepsilon_A, \eta_A)$-simulates B, and B $(\Delta_B, \varepsilon_B, \eta_B)$-simulates C, then A $(\Delta', \varepsilon', \eta')$-simulates C with

$$\varepsilon' = \varepsilon_A + \varepsilon_B + O\left(\frac{\varepsilon_A \|C\|}{\Delta_B - \|C\| + \varepsilon_B}\right)$$

$$\eta' = \eta_A + \eta_B + O\left(\frac{\varepsilon_A}{\Delta_B - \|C\| + \varepsilon_B}\right)$$

as long as $\varepsilon_A, \varepsilon_B \leq \|C\|$
and $\Delta_B \geq \|C\| + 2\varepsilon_A + \varepsilon_B$.

(Proof is not entirely straightforward.)

Let H' be a $(\Delta, \varepsilon, \eta)$-simulation of H.

Lemma (spectrum)

$\lambda_i(H) :=$ ordered eigvals of H.

$|\lambda_i(H') - \lambda_i(H)| \leq \varepsilon.$

Lemma (partition function)

Partition function $Z_H(\beta) := \operatorname{tr} e^{-\beta H}$

$$\frac{|Z_{H'}(\beta) - (p+q) Z_H(\beta)|}{(p+q) Z_H(\beta)} \leq O(e^{-\beta\Delta} + e^{\varepsilon\beta} - 1)$$

Lemma (time dynamics)

$$\| e^{-iH't} \mathcal{E}_{state}(\rho) e^{iH't} - \mathcal{E}_{state}(e^{-iHt} \rho e^{iHt}) \| \leq 2\varepsilon t + 4\eta.$$

Lemma (errors & noise)

Assume $\operatorname{rank}(P) = 1$ for projector P in encoding for H'.

(This will hold for all simulations we construct.)

Let $\mathcal{N}'$ be a channel acting on $\leq l$ qudits of H'.

$\exists \mathcal{N}$ acting on $\leq l$ qudits of H s.t.

$$\| \mathcal{N}'(\mathcal{E}_{state}(\rho)) - \mathcal{E}_{state}(\mathcal{N}(\rho)) \|_1 \leq \sqrt{\delta(4 - 3\delta)} + 8\eta.$$

This last lemma is significant. It shows that if the simulator is hit by local noise at some rate, then the result is a simulation of a noisy version of the original Hamiltonian also subject to <u>local</u> noise at a similar rate.

Since the real world is never perfect and noise-free, and the physical Hamiltonians we write down are anyway only approximations to reality, this result gives some rigorous justification to the argument that analogue Hamiltonian simulation doesn't require error correction or fault-tolerance to be useful.