Local Encodings in a Subspace

So far we have required the entire physics of the initial and target system to match identically.

On the other hand, we have not said anything about locality on the target system of the image of local observables on the initial system.

It's meaningful to relax the requirement slightly, and require all the physics to be replecated within a specified subspace of the full target Hilbert space.

This is easy, and leads directly to:

Def (encoding in a subspace)

$$\mathcal{E}(M) = V\left(M^{\oplus p} \oplus \bar{M}^{\oplus q}\right) V^{\dagger}$$

where V is an isometry (instead of having to be a full unitary).

Note:

$$\mathcal{E}(\mathbb{1}) = V^{\dagger} V = \text{projector onto the support of all } \mathcal{E}(M).$$

Note: any quantum error correcting code is such a subspace encoding, with V the encoding isometry of the code.

More interesting is when we add locality-preservation requirements.

__Def.__ (Local encoding in a subspace)

Encoding in a subspace

$$\mathcal{E}: \mathrm{Herm}\left(\bigotimes_{i=1}^{n} \mathcal{H}_i\right) \hookrightarrow \mathrm{Herm}\left(\bigotimes_{i=1}^{n} \mathcal{H}_j'\right)$$

Locality-preserving:

$$\forall A_i \in \mathcal{B}(\mathcal{H}_i) \quad \exists A_i' \in \mathcal{B}(\mathcal{H}_i') \text{ s.t.}$$

$$\mathcal{E}(A_i \otimes \mathbb{1}) = (A_i' \otimes \mathbb{1})\, \mathcal{E}(\mathbb{1})$$

why not $\mathcal{E}(\mathbb{1})(A_i' \otimes \mathbb{1})\, \mathcal{E}(\mathbb{1})$?

Doesn't make any difference:

__Lemma__

$\forall A_i \in \mathrm{Herm}(\mathcal{H}_i)$:

$$(A_i' \otimes \mathbb{1})\, \mathcal{E}(\mathbb{1}) = \mathcal{E}(\mathbb{1})(A_i' \otimes \mathbb{1}) = \mathcal{E}(\mathbb{1})(A_i' \otimes \mathbb{1})\, \mathcal{E}(\mathbb{1})$$

__Pf__

$$(A_i' \otimes \mathbb{1})\, \mathcal{E}(\mathbb{1}) = \mathcal{E}(A_i \otimes \mathbb{1}) = \mathcal{E}(A_i \otimes \mathbb{1})^\dagger$$

A_i Hermitian

$\mathcal{E}$ Hermiticity-preserving

$$= \left((A_i' \otimes \mathbb{1})\, \mathcal{E}(\mathbb{1})\right)^\dagger = \mathcal{E}(\mathbb{1})(A_i' \otimes \mathbb{1}).$$

Now recall $\mathcal{E}(\mathbb{1})$ is projector, so $\mathcal{E}(\mathbb{1})^2 = \mathcal{E}(\mathbb{1})$

□

It turns out any such local encoding is equivalent to a "tensor product" of individual encodings of each component separately:

Thm.

$\mathcal{E}$ is local iff $\exists$ encodings $\varphi_i : \mathrm{Herm}(\mathcal{H}_i) \to \mathrm{Herm}(\mathcal{H}_i')$ s.t.

$$\mathcal{E}\left(\bigotimes_{i=1}^{n} A_i\right) = \left(\bigotimes_{i=1}^{n} \varphi_i(A_i)\right) \mathcal{E}(\mathbb{1})$$

Pf.

"if": easy.

"only if":

Let $S_i^\perp \subset \mathcal{H}_i$ be subspace annihilated by $\mathcal{E}(\mathbb{1})$, i.e.

$$S_i^\perp := \mathrm{span}\{|\psi_i\rangle : \mathcal{E}(\mathbb{1})\,|\psi_i\rangle \otimes \mathbb{1} = 0\}$$

Take $\varphi_i(A_i) := \Pi_i A_i' \Pi_i$.
where Π_i = projector onto S_i.

Note:

$$\begin{aligned}(\Pi_i \otimes \mathbb{1})\,\mathcal{E}(\mathbb{1}) &= ((\mathbb{1} - \Pi_i^\perp) \otimes \mathbb{1})\,\mathcal{E}(\mathbb{1}) \\ &= (\mathbb{1} \otimes \mathbb{1}) \cdot \mathcal{E}(\mathbb{1}) = \mathcal{E}(\mathbb{1}),\end{aligned}$$

and similarly

$$\mathcal{E}(\mathbb{1})(\Pi_i \otimes \mathbb{1}) = \mathcal{E}(\mathbb{1}).$$

So

$$\mathcal{E}(A_i \otimes \mathbb{1}) = \mathcal{E}(\mathbb{1})(A_i' \otimes \mathbb{1})\,\mathcal{E}(\mathbb{1}) \qquad \text{(Lemma)}$$

$$= \mathcal{E}(\mathbb{1})(\Pi_i \otimes \mathbb{1})(A_i' \otimes \mathbb{1})(\Pi_i \otimes \mathbb{1})\,\mathcal{E}(\mathbb{1})$$

$$= (\Pi_i \otimes \mathbb{1})(A_i' \otimes \mathbb{1})(\Pi_i \otimes \mathbb{1})\,\mathcal{E}(\mathbb{1})$$

$$= (\Pi_i A_i' \Pi_i \otimes \mathbb{1})\,\mathcal{E}(\mathbb{1}) \qquad \text{(Note)}$$

$$= (\varphi(A_i) \otimes \mathbb{1})\,\mathcal{E}(\mathbb{1})$$

Similarly

$$\mathcal{E}(A_i \otimes \mathbb{1}) = \mathcal{E}(\mathbb{1})(\varphi(A_i) \otimes \mathbb{1})$$

so $[\varphi(A_i) \otimes \mathbb{1},\ \mathcal{E}(\mathbb{1})] = 0$.

Thus

$$\mathcal{E}\Big(\bigotimes_i A_i\Big) = \mathcal{E}\Big(\prod_i (A_i \otimes \mathbb{1})\Big)$$

$$= \prod_i \mathcal{E}(A_i \otimes \mathbb{1}) \qquad \mathcal{E} \text{ encoding}$$

$$= \prod_i \big[(\varphi(A_i) \otimes \mathbb{1})\,\mathcal{E}(\mathbb{1})\big]$$

$$= \Big(\prod_i (\varphi(A_i) \otimes \mathbb{1})\Big)\,\mathcal{E}(\mathbb{1}) \qquad \text{commutes, } \mathcal{E}(\mathbb{1}) \text{ projector}$$

$$= \Big(\bigotimes_i \varphi(A_i)\Big)\,\mathcal{E}(\mathbb{1})$$

as required.

Remains to show φ_i are encodings.

- $\varphi_i(A_i)$ clearly Hermitian
- $\varphi_i(A_i) \otimes \mathbb{1}$ commutes with projector $\mathcal{E}(\mathbb{1})$

$\Rightarrow$ $\operatorname{spec}(\varphi_i(A_i) \otimes \mathbb{1})\,\mathcal{E}(\mathbb{1})) \subseteq \operatorname{spec}(\varphi_i(A_i) \otimes \mathbb{1})$

$$\mathcal{E}(\mathbb{1}) = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$\begin{array}{cc} \varphi_i(A_i) \otimes \mathbb{1} = \left(\begin{array}{c|c} \lambda_i \ \lambda_i \ \ddots & 0 \\ \hline 0 & \lambda_i \ \lambda_i \ \ddots \ 0 \ 0 \end{array}\right) & \begin{array}{c} |\psi\rangle|\phi\rangle \\ \\ |\psi\rangle|\phi'\rangle \end{array} \end{array}$$

$\varphi_i(A_i) = \Pi_i A' \Pi_i \Rightarrow \operatorname{supp} \varphi_i(A_i) \subseteq S_i$

Consider eigvect $|\psi\rangle \in S_i$ of $\varphi_i(A_i)$ with associated eigval λ.

$|\psi\rangle \otimes |\phi\rangle$ simultaneous eigvect of $\varphi_i(A_i) \otimes \mathbb{1}$ & $\mathcal{E}(\mathbb{1})$

$\Rightarrow$ $\mathcal{E}(\mathbb{1})\,|\psi\rangle|\phi\rangle = 0$ or $|\psi\rangle|\phi\rangle$.

If $\mathcal{E}(\mathbb{1})|\psi\rangle|\phi\rangle$ for all $|\phi\rangle$

$\Rightarrow \mathcal{E}(\mathbb{1})|\psi\rangle \otimes \mathbb{1} = 0$

$\Rightarrow |\psi\rangle \in S_i^{\perp}$ contradiction

$\exists\, |\phi\rangle$ s.t. $\mathcal{E}(\mathbb{1})|\psi\rangle|\phi\rangle = |\psi\rangle|\phi\rangle$

$\Rightarrow \lambda$ is also an eigval of $(\varphi_i(A_i)\otimes\mathbb{1})\,\mathcal{E}(\mathbb{1})$

$\Rightarrow \operatorname{spec}(\varphi_i(A_i)\otimes\mathbb{1}|_{S_i}) \subseteq \operatorname{spec}((\varphi_i(A_i)\otimes\mathbb{1})\,\mathcal{E}(\mathbb{1})|_{\mathcal{E}(\mathbb{1})})$

Thus $\operatorname{spec}((\varphi_i(A_i)\otimes\mathbb{1})\,\mathcal{E}(\mathbb{1})) = \operatorname{spec}(\varphi_i(A_i)\otimes\mathbb{1}|_{S_i})$

$$\begin{aligned}
\operatorname{spec}(\varphi_i(A_i)|_{S_i}) &= \operatorname{spec}(\varphi_i(A_i)\otimes\mathbb{1}|_{S_i}) \\
&= \operatorname{spec}((\varphi_i(A_i)\otimes\mathbb{1})\,\mathcal{E}(\mathbb{1})|_{\mathcal{E}(\mathbb{1})}) \\
&= \operatorname{spec}(\mathcal{E}(A_j\otimes\mathbb{1})|_{\mathcal{E}(\mathbb{1})}) \\
&= \operatorname{spec}(A_i)
\end{aligned}$$

φ_i spectrum preserving.

- $$\begin{aligned}
\varphi_i(\lambda A_i + \mu B_i)\,\mathcal{E}(\mathbb{1}) &= \mathcal{E}((\lambda A_i + \mu B_i)\otimes\mathbb{1}) \\
&= \lambda\,\mathcal{E}(A_i\otimes\mathbb{1}) + \mu\,\mathcal{E}(B_i\otimes\mathbb{1}) \qquad \mathcal{E} \text{ encoding} \\
&= [(\lambda\,\varphi_i(A_i) + \mu\,\varphi_i(B_i))\otimes\mathbb{1}]\,\mathcal{E}(\mathbb{1})
\end{aligned}$$

$$\Rightarrow \underbrace{[\varphi_i(\lambda A_i + \mu B_i) - \lambda\,\varphi_i(A_i) + \mu\,\varphi_i(B_i)]}_{M}\otimes\mathbb{1}\cdot\mathcal{E}(\mathbb{1}) = 0$$

φ_i real-linear if we can show $M = 0$.

But $M \otimes \mathbb{1}$ commutes with $\mathcal{E}(\mathbb{1})$ whilst M has no support on $S_i^\perp$.

Similar argument by simultaneous eigvects as above $\Rightarrow M \otimes \mathbb{1} = 0 \Rightarrow M = 0$

φ_i real-linear.

$\therefore$ φ_i is an encoding, as claimed.

$\square$

Thm.

$\mathcal{E} : \mathcal{B}(\bigotimes_{i=1}^{n} \mathcal{H}_i) \longrightarrow \mathcal{B}(\bigotimes_{i=1}^{n} \mathcal{H}_i')$ is a local encoding iff

$$\mathcal{E}(M) = V(M \otimes P + \bar{M} \otimes Q)V^\dagger$$

where

- $V = \bigotimes_{i=1}^{n} V_i$ for isometries $V_i : \mathcal{H}_i \otimes E_i \rightarrow \mathcal{H}_i'$
- P, Q orthogonal projectors on $E := \bigotimes_i E_i$

$\forall i \; \exists$ orthogonal projectors P_i, Q_i on E_i s.t.

$$(P_i \otimes \mathbb{1})P = P \quad , \quad (Q_i \otimes \mathbb{1})Q = Q$$

We say P, Q are "locally distinguishable".

Pf.

"if": easy

"only if":

We have two ways to characterise local encodings now:

As an encoding in a subspace:

$$\begin{aligned}
\mathcal{E}(A_i \otimes \mathbb{1}) &= W(A_i \otimes \mathbb{1} \otimes \tilde{P} + \bar{A}_i \otimes \mathbb{1} \otimes \tilde{Q})W^\dagger \\
&= W(A_i \otimes \mathbb{1})W^\dagger W(\mathbb{1} \otimes \tilde{P})W^\dagger \\
&\quad + W(\bar{A}_i \otimes \mathbb{1})W^\dagger W(\mathbb{1} \otimes \tilde{Q})W^\dagger \\
&=: W(A_i \otimes \mathbb{1})W^\dagger P' + W(\bar{A}_i \otimes \mathbb{1})W^\dagger Q'
\end{aligned}$$

$$\text{where} \quad \left.\begin{array}{l} P' : W(\mathbb{1} \otimes \tilde{P})W^\dagger \\ Q' : W(\mathbb{1} \otimes \tilde{Q})W^\dagger \end{array}\right\} \text{orthog} \qquad (1)$$

As a "tensor product" of encodings (Thm):

$$\varphi_i(A_i) = V_i(A_i \otimes P_i + \bar{A}_i \otimes Q_i)V^\dagger \quad \text{encoding}$$

isometry $V_i : \mathcal{H}_i \otimes E_i \to \mathcal{H}_i'$

Let $V := \bigotimes_i V_i$

$$\mathcal{E}(A_i \otimes \mathbb{1}) = \Big(\varphi_i(A_i) \otimes \bigotimes_{j \neq i} \varphi_j(\mathbb{1})\Big)\, \mathcal{E}(\mathbb{1})$$

$$= V(A_i \otimes P_i \otimes \mathbb{1} + A_i \otimes Q_i \otimes \mathbb{1})V^\dagger \cdot W(\mathbb{1} \otimes \hat{P} + \mathbb{1} \otimes \hat{Q})W^\dagger$$

$$= V(A_i \otimes P_i \otimes \mathbb{1} + A_i \otimes Q_i \otimes \mathbb{1})V^\dagger \cdot (P' + Q')$$

$$= V(A_i \otimes \mathbb{1})V^\dagger V(\mathbb{1} \otimes P_i)V^\dagger (P' + Q') + V(\bar{A}_i \otimes \mathbb{1})V^\dagger V(\mathbb{1} \otimes Q_i)V^\dagger (P' + Q')$$

$$\begin{aligned} =: \; & V(A_i \otimes \mathbb{1})V^\dagger \tilde{P}_i \; P' \\ & + V(A_i \otimes \mathbb{1})V^\dagger \tilde{P}_i \; Q' \\ & + V(\bar{A}_i \otimes \mathbb{1})V^\dagger \tilde{Q}_i \; P' \\ & + V(\bar{A}_i \otimes \mathbb{1})V^\dagger \tilde{Q}_i \; Q' \end{aligned} \qquad (2)$$

where $\tilde{P}_i := V(\mathbb{1} \otimes P_i)V^\dagger$, $\tilde{Q}_i := V(\mathbb{1} \otimes Q_i)V^\dagger$

Applying these to $A_i = i\mathbb{1}$:

(1) $\Rightarrow$

$$\mathcal{E}(i\mathbb{1}) = W\, i\mathbb{1}\, W^\dagger P' - W\, i\mathbb{1}\, W^\dagger Q'$$

$$= i P' - i Q'$$

(2) $\Rightarrow$

$$\mathcal{E}(i\mathbb{1}) = i\,\tilde{P}_i P' + i\,\tilde{P}_i Q' - i\,\tilde{Q}_i P' - i\,\tilde{Q}_i P'$$

Identifying $\pm i$ eigenspaces & recalling $P'Q'=0$ gives:

$$P' = \tilde{P}_i P', \quad Q' = \tilde{Q}_i Q', \quad \tilde{Q}_i P' = \tilde{P}_i Q' = 0$$

Thus (1) & (2) $\Rightarrow$

$$W(A_i \otimes \mathbb{1})W^\dagger P' = V(A_i \otimes \mathbb{1})V^\dagger \tilde{P}_i P'$$

$$V^\dagger W (A_i \otimes \mathbb{1}) W^\dagger W (\mathbb{1} \otimes \tilde{P}) W^\dagger$$
$$= V(A_i \otimes \mathbb{1}) V^\dagger W (\mathbb{1} \otimes \tilde{P}) W^\dagger$$

using $\tilde{P}_i P' = P'$, $P' := W(\mathbb{1} \otimes \tilde{P})W^\dagger$

$$\underbrace{V^\dagger W(\mathbb{1} \otimes \tilde{P})}_{=: M}(A_i \otimes \mathbb{1}) = (A_i \otimes \mathbb{1})\underbrace{V^\dagger W(\mathbb{1} \otimes \tilde{P})}_{=: M}$$

$V^\dagger V = W^\dagger W = \mathbb{1}$ isometries

$$M(A_i \otimes \mathbb{1}) = (A_i \otimes \mathbb{1})M.$$

Let $M = \sum_k B_k \otimes C_k$ be operator Schmidt decomposition of M.

$$\Rightarrow \sum_k [A_i, B_k] \otimes C_k = 0$$

$\Rightarrow \forall A_i \in \mathcal{M}_n : \; [A_i, B_k] = 0$
using linear independence of $B_k \otimes C_k$.

Schur's Lemma

$$\forall A \in \mathcal{M}_n : [A, X] = 0 \Rightarrow X \propto \mathbb{1}.$$

Thus we have $\forall_k \; B_k \propto \mathbb{1}$

$$\Rightarrow V^\dagger W(\mathbb{1} \otimes \tilde{P}) = \mathcal{M} = \sum_k B_k \otimes C_k$$

$$= \mathbb{1} \otimes \tilde{V} \tilde{P}$$

for some isometry $\tilde{V}$.

Similarly, $V^\dagger W(\mathbb{1} \otimes \tilde{Q}) = \mathbb{1} \otimes \tilde{V} \tilde{Q}$

Take $P := \tilde{V} \tilde{P} \tilde{V}^\dagger$, $Q := \tilde{V} \tilde{Q} \tilde{V}^\dagger$.

Then

$$\mathcal{E}(\mathcal{M}) = \mathcal{E}(\mathbb{1})\, \mathcal{E}(\mathcal{M})\, \mathcal{E}(\mathbb{1})$$

$$= V V^\dagger W(\mathcal{M} \otimes \tilde{P} + \bar{\mathcal{M}} \otimes \tilde{Q}) W^\dagger V V^\dagger$$

from (1)

$$= V \underbrace{V^\dagger W(\mathbb{1} \otimes \tilde{P})}_{= \mathbb{1} \otimes \tilde{V} \tilde{P}} (\mathcal{M} \otimes \mathbb{1}) \underbrace{(\mathbb{1} \otimes \tilde{P}) W^\dagger V}_{= \mathbb{1} \otimes \tilde{P} \tilde{V}^\dagger} V^\dagger$$

$$+ V \underbrace{V^\dagger W(\mathbb{1} \otimes \tilde{Q})}_{= \mathbb{1} \otimes \tilde{V} \tilde{Q}} (\bar{\mathcal{M}} \otimes \mathbb{1}) \underbrace{(\mathbb{1} \otimes \tilde{Q}) W^\dagger V}_{= \mathbb{1} \otimes \tilde{Q} \tilde{V}^\dagger} V^\dagger$$

$$= V(\mathcal{M} \otimes \tilde{V} \tilde{P} \tilde{V}^\dagger + \bar{\mathcal{M}} \otimes \tilde{V} \tilde{Q} \tilde{V}^\dagger) V^\dagger$$

$$= V(\mathcal{M} \otimes P + \bar{\mathcal{M}} \otimes Q) V^\dagger$$

where $V := \bigotimes_i V_i$

Finally,

$$\tilde{P}_i P' = P'$$

$$V(\mathbb{1} \otimes P_i)V^\dagger W(\mathbb{1} \otimes \tilde{P})W^\dagger = W(\mathbb{1} \otimes \tilde{P})W^\dagger$$

$$(\mathbb{1} \otimes P_i) V^\dagger W (\mathbb{1} \otimes \tilde{P}) = V^\dagger W (\mathbb{1} \otimes \tilde{P})$$

$$(\mathbb{1} \otimes P_i)(\mathbb{1} \otimes \tilde{V}\tilde{P}) = (\mathbb{1} \otimes \tilde{V}\tilde{P})$$

$$(\mathbb{1} \otimes P_i)(\mathbb{1} \otimes \tilde{V}\tilde{P})(\mathbb{1} \otimes \tilde{P}\tilde{V}^\dagger) = (\mathbb{1} \otimes \tilde{V}\tilde{P})(\mathbb{1} \otimes \tilde{P}\tilde{V}^\dagger)$$

$$\Rightarrow (\mathbb{1} \otimes P_i) P = P.$$

Similarly, $\tilde{Q}_i Q' = Q'$, $\tilde{P}_i Q' = \tilde{Q}_i P' = 0$

$$\Rightarrow (\mathbb{1} \otimes Q_i) Q = Q$$

$$(\mathbb{1} \otimes P_i) Q = (\mathbb{1} \otimes Q_i) P = 0$$

so P & Q are locally distinguishable as required.

$\square$

Schematically, Thm. implies all local encodings look like this:

$$\begin{array}{ccc}
\mathcal{H}_1 \;\; E_1 & \xrightarrow{V_1} & \mathcal{H}_1' \\
\mathcal{H}_2 \;\; E_2 & \xrightarrow{V_2} & \mathcal{H}_2' \\
\vdots & & \\
\mathcal{H}_n \;\; E_n & \xrightarrow{V_n} & \mathcal{H}_n'
\end{array}$$