TNS. A global perspective

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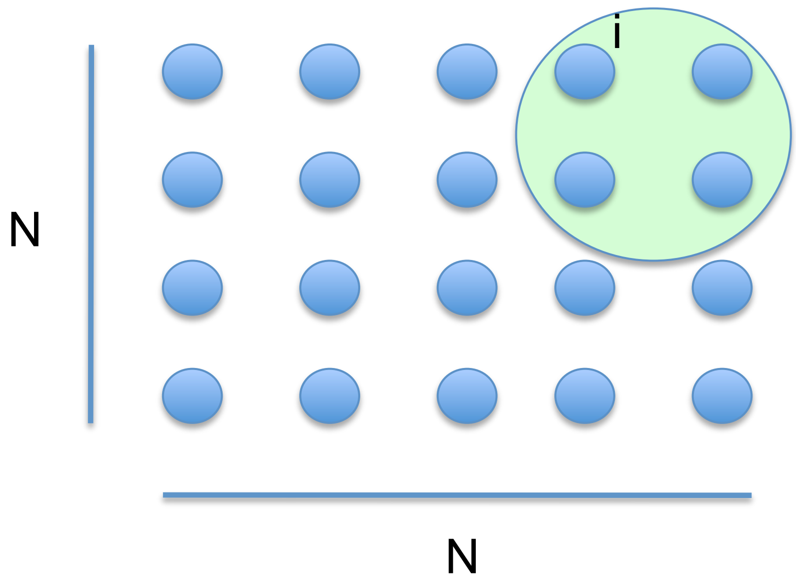
SETUP



Particles in a lattice

A space $H_i = C^d$ associated to each particle

Space of the joint system = tensor product

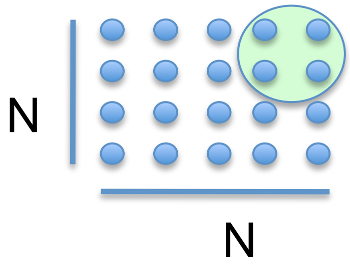
$$= \bigotimes_i H_i \cong C^{d^{N^2}}$$

Particles interact with those nearby in a uniform way h = hermitian matrix of small size $(d^r \times d^r)$, r the number of nearby particles).

Huge!!

h_i matrix h located at position i.

Hamiltonian $H = \sum_i h_i \otimes 1_{rest}$ matrix of size $d^{N^2} \times d^{N^2}$



SETUP

Normalized vectors in $\bigotimes_i H_i$ are the ***states*** of the system (encode all properties)

Hamiltonian H = Energy observable : The energy of a state is $\langle \psi | H | \psi \rangle$

Energy levels of the system = eigenvalues of H.

States with minimal energy = eigenvector of $\lambda_0(N)$
Called ***ground states***.

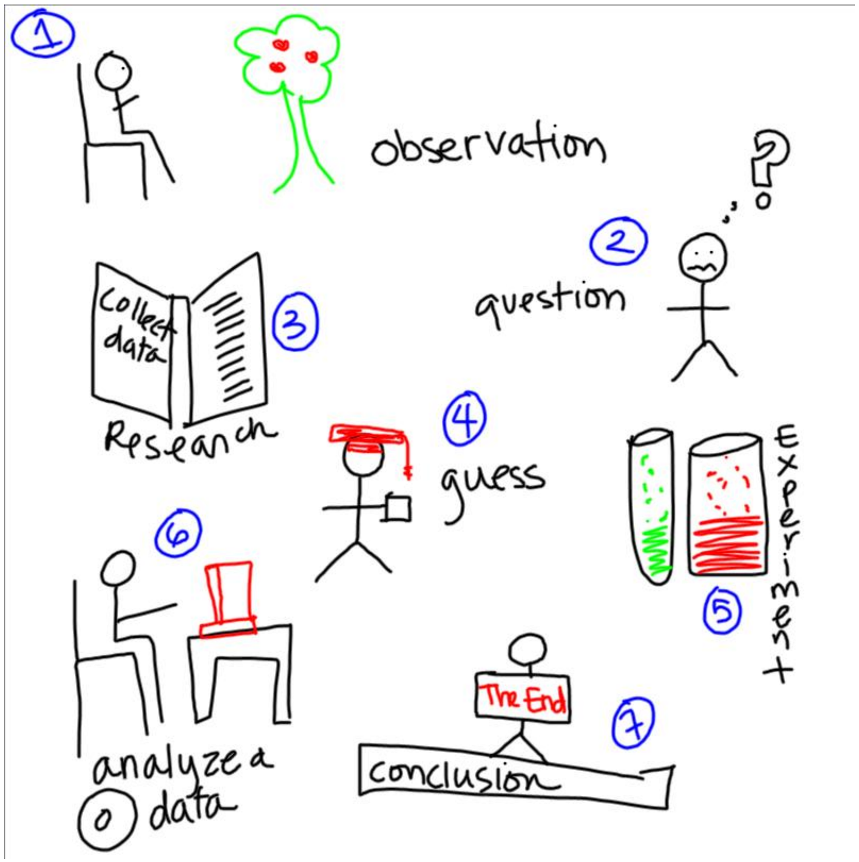
Eigenstates of $\lambda_1(N)$ called elementary ***excitations***.

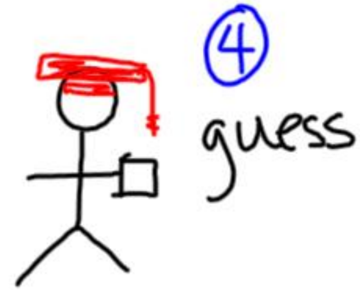
Spectral Gap: $\Delta_N = \lambda_1(N) - \lambda_0(N)$
Energy to pay to jump from ground to excited states

Eigenvectors are stable states since the evolution eq. is

$$\frac{\partial v(t)}{\partial t} = -iHv(t)$$

Back to school. The scientific method





Two steps:

1.- Model the interactions in the system = Hamiltonian

2.- Give predictions for the observable quantities

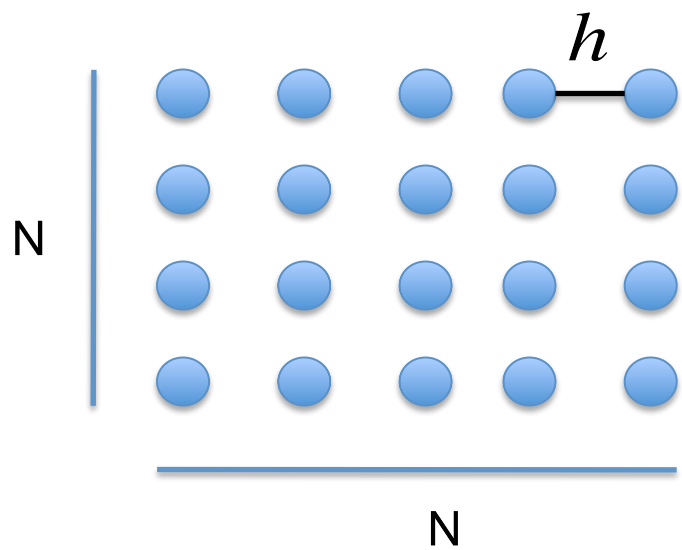
Find a good (efficient) description of the ground state which allow to compute such observable quantities and (optimally) help in understanding the physics of the system

TNS

The good description. Does it exist?

Find a good (efficient) description of the ground state which allow to compute such observable quantities and (optimally) help in understanding the physics of the system

In principle yes (counting parameters):



$$H = \sum_i h_i \otimes 1_{rest}$$

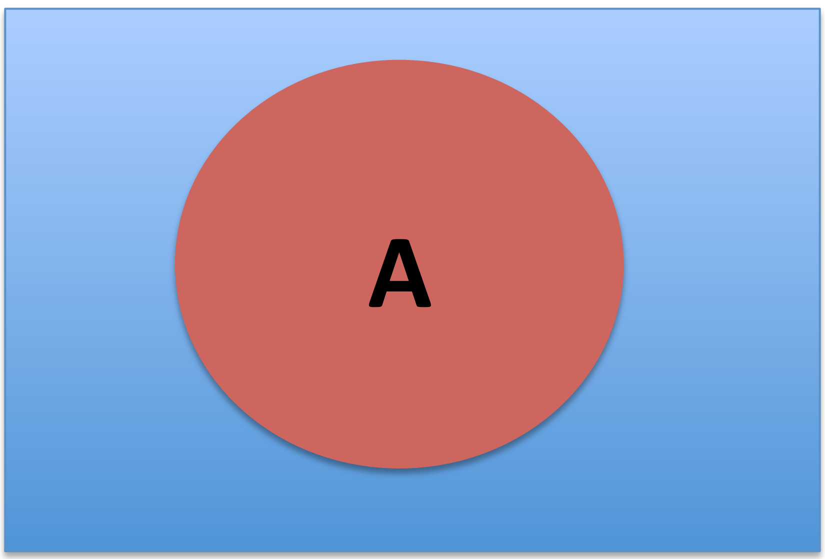
Hamiltonian: Number of parameters independent of system size.

$$\bigotimes_i H_i \cong C^{d^{N^2}}$$

Hilbert space: exponentially big

The good description. How does it look like? The area law.

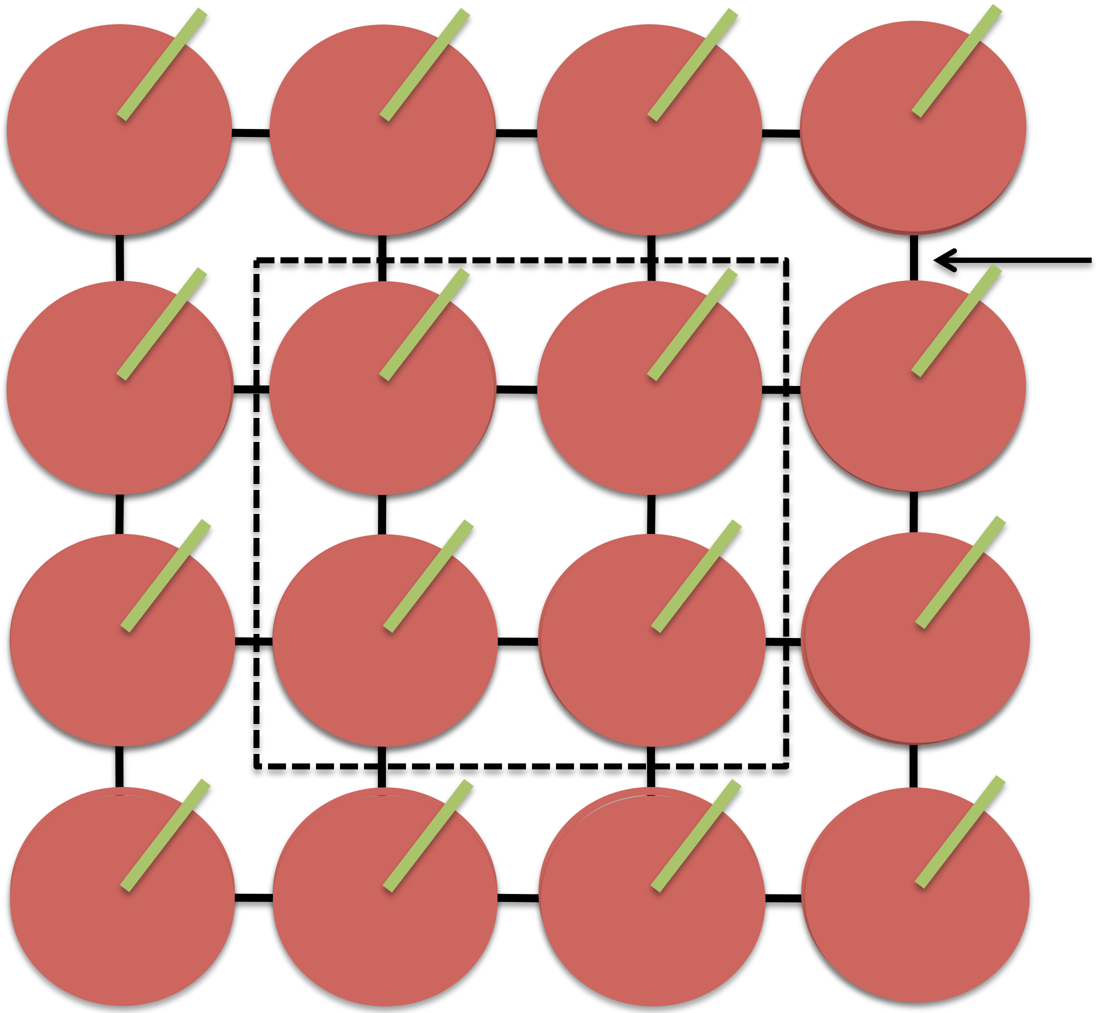
Find a good (efficient) description of the ground state which allow to compute such observable quantities and (optimally) help in understanding the physics of the system



Ground states of local gapped Hamiltonians verify the AREA LAW

$$S(\rho_A) \leq c|\partial A|$$

The good description. How to enforce the area law. A guess



$$\sum_{i=1}^{D} |ii\rangle$$

D = bond dimension

PEPS

MPS

The good description. It was a good guess!

Theorem (Hastings 2007, Arad-Landau-Kitaev-Vazirani 2013):
In 1D, if a Hamiltonian is local and has a gap, then the bond dimension of the MPS tensor required to approximate the ground state of H within error E in a system of size N scales sublinearly with N/E.

Theorem (Hastings 2006, Molnar-Schuch-Verstraete-Cirac 2014):
If a Hamiltonian H has a gap and, for each energy E, the density of states with energy smaller than E scales only polynomially with the system size, then the size of the PEPS tensor required to approximate the ground state of H within error E in a system of size N scales quasi-polynomially with N/E

The good description. Does it exist?

Find a good (efficient) description of the ground state which allow to compute such observable quantities and (optimally) help in understanding the physics of the system

Natural questions:

Given a Hamiltonian. How to find the PEPS approximation to the ground state? How to compute quantities from it?

Numerical methods (DMRG, ITBD, TDVP, IPEPS …).
NOT COVERED IN THIS COURSE.

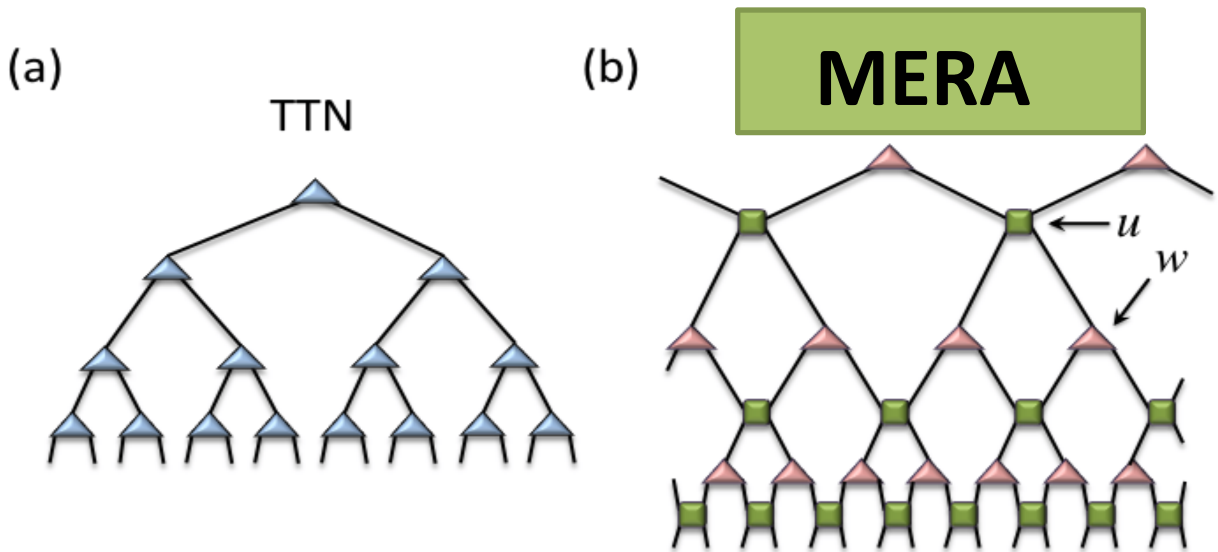
Given a PEPS. How to understand the physics of the system.

THIS COURSE

Is this all?

NO! We did not talk yet about GAPLESS Hamiltonians. That is, about PHASE TRANSITIONS.

In phase transitions one expects a self-similar behavior which is captures by the following type of TNS:



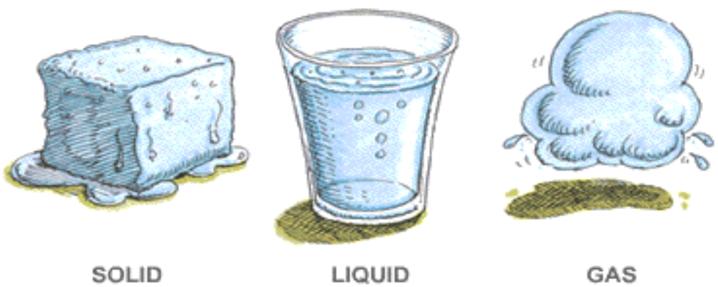
NOT COVERED IN THIS COURSE

Tensor Network States and the Classification of Quantum Phases

Quantum phases

What is a phase?

Temperature

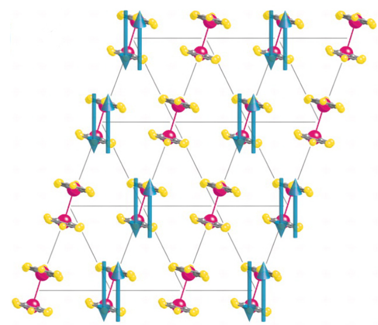


Phase transitions

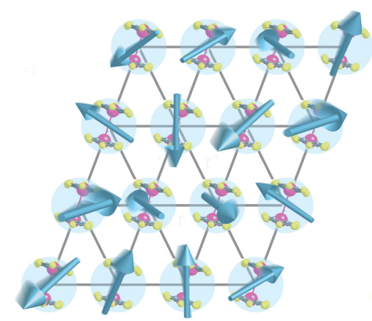
At zero temperature: Quantum phases.

They include very **exotic phenomena:** topological dependency, superconductivity, spin liquids, etc.

Strength of repulsion terms



Crystal



Spin liquid

A **phase** should be something like: “the equivalence class of all states of matter with *similar* properties”

Quantum phases

PHASE = an equivalence relation on the set of finite range interactions $\bigcup_r M_{d^r}$

Two systems governed by interactions $h^0, h^1 \in M_{d^r}$ are in the same phase iff there is a smooth path of interactions $[0,1] \ni \alpha \mapsto h^\alpha$ and a constant c>0 s.t the gap $\Delta_N(\alpha)$ of the Hamiltonian $H_\alpha = \sum_i h_i^\alpha \otimes 1_{rest}$ is $\Delta_N(\alpha) > c$ for all N, α.

Two main reasons for this definition:

It is stable against small errors in the interactions.

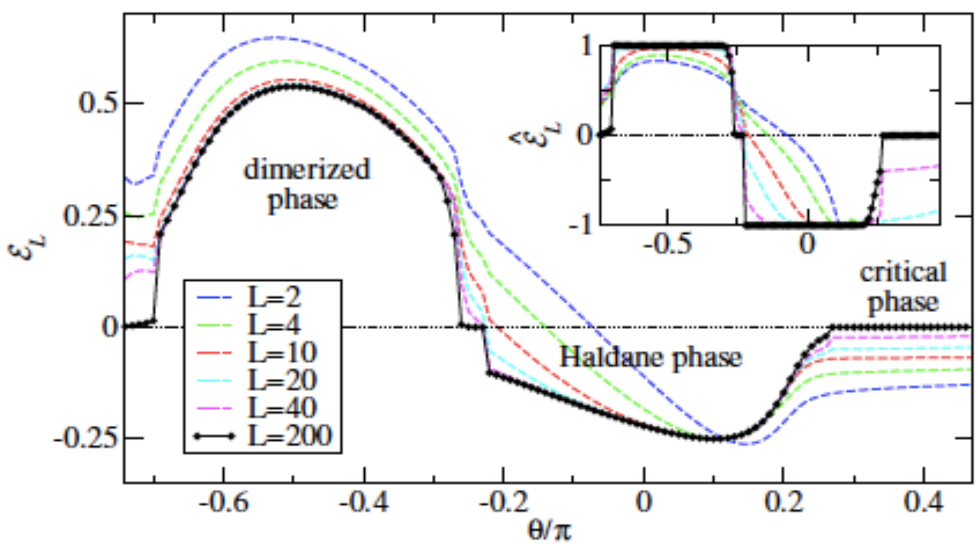
Observable quantities on the ground state behave smoothly through the path (no phase transitions).

THE AIM. DIFFERENT THAN USUAL

Usual approach to quantum phases

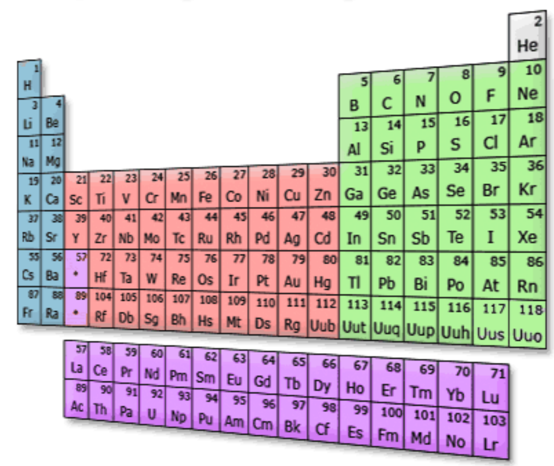
Phase diagram of a particular parametrized model

$$H(\theta) = \cos(\theta)\sum_i S_i \cdot S_{i+1} + \sin(\theta)\sum_i (S_i \cdot S_{i+1})^2$$



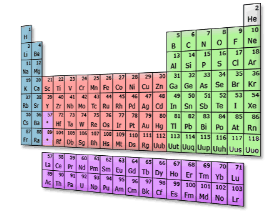
THE AIM

Equivalence relation in the set of ALL possible interactions.



Periodic table of locally interacting quantum spin systems

Periodic table in 1D. Ingredients

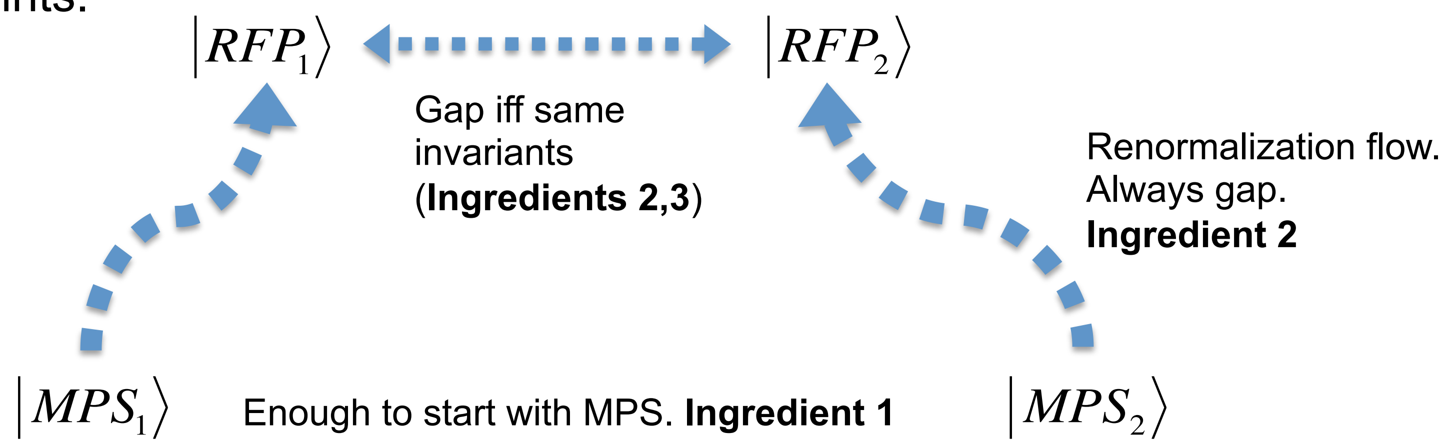


Ingredient 1 (Hastings 2007, Arad et al.): Matrix Product States (MPS) approximate well ground states of Hamiltonians

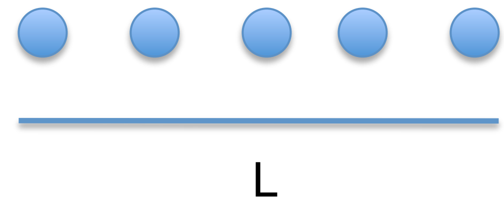
From interactions to states

Ingredient 2 (Nachtergaele 1995): A lower bound on the spectral gap of certain (parent) Hamiltonians having MPS as ground states.

Ingredient 3 (Verstraete et al. 2005, Perez-Garcia et al. 2007): A good description of renormalization transformations in MPS and the structure (phase invariants) of their fixed points.



Matrix Product States/Operators



A bit more of detail.

$$\left|MPS_A\right\rangle = \sum_{i_1 i_2 \ldots i_L = 1}^{d} tr(A_{i_1} A_{i_2} \ldots A_{i_L}) e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_L}$$

A_i : Matrices of size D

$$\rho_{MPO_A} = \sum_{i_1 i_2 \ldots i_L, j_1 j_2 \ldots j_L = 1}^{d} tr(A_{i_1, j_1} \ldots A_{i_L, j_L}) E_{i_1, j_1} \otimes \ldots \otimes E_{i_L, j_L}$$

$A_{i,j}$: Matrices of size D

Fundamental Theorem of MPS/MPO

PG, Wolf, Verstraete, Cirac QIC 2007 + IEEE Trans. Inf. Theory 2010 + …

A_i can be diagonalized by blocks so that for all L>D^5 one gets that

$$\oplus_i (M_{D_i} \otimes \rho_i^L) = \mathrm{span}\{A_{i_1} \ldots A_{i_L} \mid i_1, \ldots, i_L = 1, \ldots, d\}$$

States associated to different blocks are asymptotically orthogonal.

Moreover, the canonical form is unique, in the sense that two MPS in canonical form

$$\left|MPS_A\right\rangle = \left|MPS_B\right\rangle$$

If and only if there exists an invertible an block-diagonal map Y so that:

$$YA_iY^{-1} = B_i \quad \forall \mathrm{i}$$

Corollary: characterization of symmetries

PG, Wolf, Sanz, Verstraete, Cirac, Phys. Rev. Lett. 2008

Let G be a compact group (associated to the symmetry present in the system) and

$$g \in G \mapsto u_g \in M_d$$

A representation of G.

The MPS given by A is invariant under the symmetry

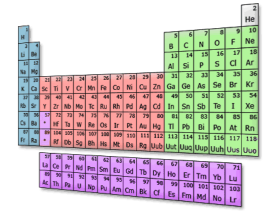
$$|MPS_A\rangle = u_g^{\otimes L} |MPS_A\rangle$$

If and only if there exist a projective representation of G so that

$$V_g A_i V_g^* = \sum_j u_g(i,j) A_j \quad \forall \mathrm{i}$$

$$H^2(G, U(1))$$

Periodic table in 1D. Solution



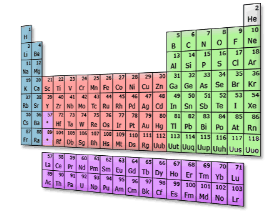
PERIODIC TABLE IN 1D without symmetries and PBC:
Phases indexed by degeneracy of ground space

PERIODIC TABLE IN 1D with symmetries and PBC:
Phases indexed by equivalent classes of projective representations

$$H^2(G, U(1))$$

Pollmann, Berg, Turner, Oshikawa, Phys. Rev. B. 81, 064439 (2010)
Chen, Gu, Wen, Phys. Rev. B 83, 035107 (2011)
Fidkowski, Kitaev, Phys. Rev. B 83, 075103 (2011)
Schuch, Pérez-García, Cirac, Phys. Rev. B 84, 165139 (2011)
Haegeman, Pérez-García, Cirac, Schuch, Phys. Rev. Lett. 109, 050402 (2012)

Boundary conditions make the classification more complicated: Ogata's talk



The generic case

Collins, González-Guillén, PG, Commun. Math. Phys. 2013

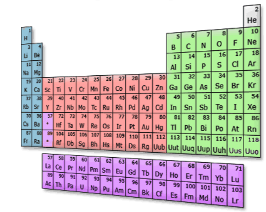
The canonical form allows to identify the set of MPS with the group U(dD). There is a natural way to sample MPS via the Haar measure in U(dD).

Theorem: Except for exponentially small probability (in D), for all observable O supported on r consecutive spins, we have

$$\left| \langle MPS_A | O | MPS_A \rangle - \frac{\mathrm{tr}(O)}{d^r} \right| \leq \frac{d^{2r}}{D^{1/10}}$$

That is, generic MPS behave as the maximally mixed state.

In 2D?



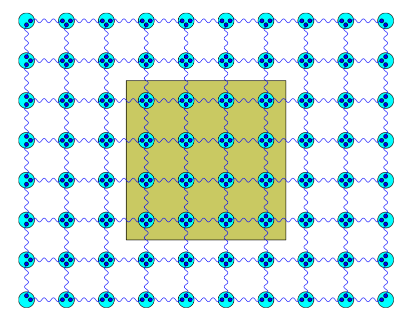
The problem is much more difficult due to the existence of topological order:

1. Dimension of the ground space depends on the topology (sphere vs. torus)
2. All ground states are locally indistinguishable
3. Elementary exictations behave as quasi-particles with exotic braiding and fusion rules.

Fortunately, we still have **Ingredient 1:**

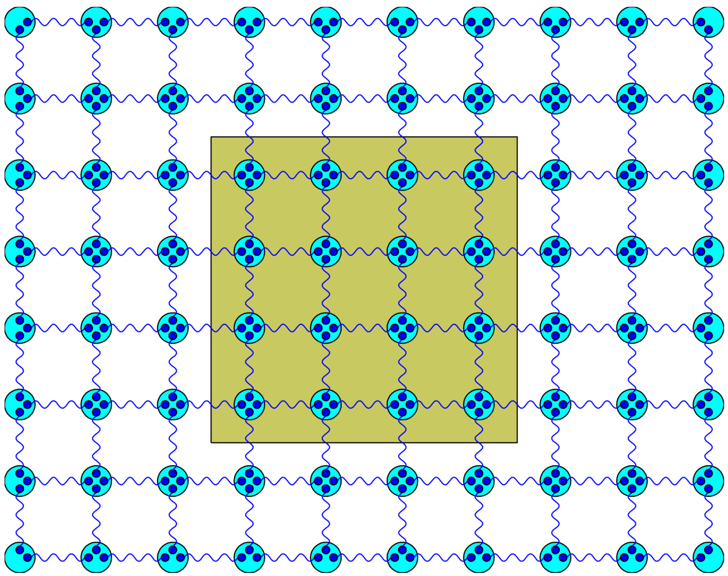
A good parametrization of ground states in 2D

(Hastings 2007, Molnar et al. 2014).



Projected Entangled Pair States (PEPS). 2D version of MPS

PEPS



$$|MES\rangle = \frac{1}{\sqrt{D}}(e_1 \otimes e_1 + \ldots + e_D \otimes e_D)$$

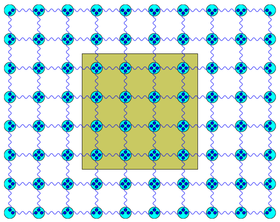
$$A = C^D \otimes C^D \otimes C^D \otimes C^D \rightarrow C^d$$

$$|PEPS_A\rangle = A^{\otimes \text{vertices}}(|MES\rangle^{\otimes \text{edges}})$$

Fundamental Theorem of PEPS

$$|PEPS_A\rangle = |PEPS_B\rangle \Leftrightarrow \exists Y, Z : A(Y \otimes Z \otimes Y^{-1} \otimes Z^{-1}) = B$$

Implies the corresponding characterization of symmetries



Topological order in PEPS

Schuch, Cirac, PG, Annals of Physics 2010

Let G be a finite group and consider the left regular representation:

$$g \in G \rightarrow L_g \in M_{D=|G|}$$

Theorem: The PEPS given by A has the topological order given by G (excitations are representations of the Hopf algebra D(G): Drienfeld Double of G) if:

$$A(L_g \otimes L_g \otimes L_g^* \otimes L_g^*) = A$$

Generalized by Verstraete’s group for a general fusion algebra

Theorem (Cirac, PG, Schuch, Verstraete, arXiv:1606.00608): Those cover all phases that contain a RFP.

Are there 2D phases without RFP? J. Haah (2014): in 3D YES.

Structure of the course

1. Basics of MPS: transfer operator, expectation values, RFP
2. The Fundamental Theorem of MPS
3. Parent Hamiltonian of MPS and its spectral gap.
4. Topological order in 2D and its characterization in PEPS

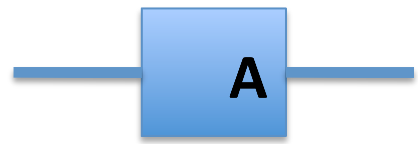
Appendix. Box-leg notation for tensors

Each leg = one index

vector

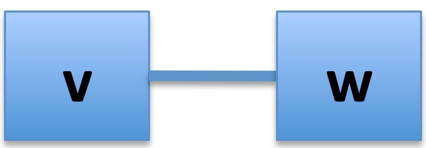
$$= \sum_i v_i |i\rangle$$

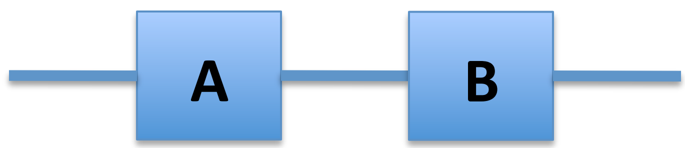
matrix



$$= \sum_{ij} A_{ij} |i\rangle |j\rangle$$

Joining leg = tensor contraction



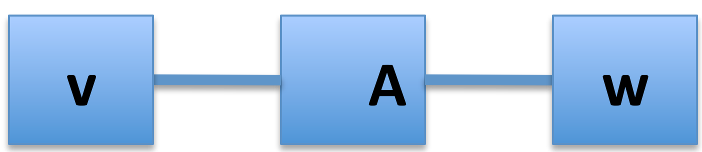


$$= \sum_{ijk} A_{ij} B_{jk} |i\rangle |k\rangle = AB$$

Scalar product

$$\sum_i v_i w_i$$

Matrix Multiplication



$$= \sum_{ijk} A_{ij} v_i w_j$$