QMA-hardness of Local Hamiltonian

Recall:

Def (QMA + amplification)
Decision (YES/NO) problem, problem size n.

Problem $\in$ QMA if
$\exists\ T = O(\text{poly}(n))$, $\varepsilon = \Omega\left(\frac{1}{\text{poly}(n)}\right)$,
quantum circuit $U = U_T \cdot U_{T-1} \cdots U_2 \cdot U_1$ s.t.

YES:
$\exists\ |w\rangle$: $\Pr(U|w\rangle$ outputs "1"$) \geq 1-\varepsilon$

NO:
$\forall\ |w\rangle$: $\Pr(U|w\rangle$ " " $) \leq \varepsilon$

Def (Local Hamiltonian Problem)

Input: k-local Hamiltonian H on n qudits with m local terms, where $\lambda_0(H) \leq \alpha$ or $\lambda_0(H) \geq \beta$ with $\beta - \alpha \geq \frac{1}{\text{poly}(n)}$

Output: YES if $\lambda_0(H) \leq \alpha$
NO if $\lambda_0(H) \geq \beta$

Thm (Kitaev)
The Local Hamiltonian problem is QMA-hard

We saw that entanglement defeats naive proof approach.

$\rightarrow$ Need smarter way of encoding witness verification computation into Hamiltonian.

__Idea__ (Feynman): encode evolution of computation in quantum __superposition__:

"Computational history state" or "history state":

$$|\Psi\rangle = \frac{1}{\sqrt{T}} \sum_{t=0}^{T} |\psi_t\rangle |t\rangle_{cl}$$

computation register $(\mathbb{C}^2)^{\otimes n}$

clock register $\mathbb{C}^{T+1}$

$|\psi_t\rangle$ = state after t steps (gates)

$|\psi_0\rangle$ = initial input state

$|\psi_T\rangle$ = final output state

Clock register takes states $\in \mathbb{C}^{T+1}$:
$|0\rangle, |1\rangle, |2\rangle, \ldots, |t\rangle, \ldots |T-1\rangle, |T\rangle$

Easy to write down (non-local) Hamiltonian with $|\Psi\rangle$ as ground state:

$$H = H_{in} + H_{prop}$$

$$= (\mathbb{1} - |\psi_0\rangle\langle\psi_0|) \otimes |0\rangle\langle 0|_{cl}$$

ensures correct initial state

$$+ \sum_{t=0}^{T-1} \Big(|\psi_t\rangle\langle\psi_t| \otimes |t\rangle\langle t| + |\psi_{t+1}\rangle\langle\psi_{t+1}| \otimes |t+1\rangle\langle t+1|$$

forces clock to "tick"

$$- |\psi_{t+1}\rangle\langle\psi_t| \otimes |t+1\rangle\langle t|$$

forces 1 step of computation & increments clock

$$- |\psi_t\rangle\langle\psi_{t+1}| \otimes |t\rangle\langle t+1| \Big)$$

necessary for Hermiticity; forces 1 step of computation <u>backwards in time</u> & <u>decrements</u> clock.

<u>Exercise 4</u>

Show that unique g.s. of H is $|\Psi\rangle$.

Challenge is to:

- construct <u>local</u> H for QMA verifier
- prove YES instance $\Rightarrow \lambda_0(H) \leq \alpha$
- prove NO instance $\Rightarrow \lambda_0(H) \geq \beta$

$\left(\beta - \alpha \geq \frac{1}{\mathrm{poly}(n)}\right.$ promise gap $\left.\right)$

<u>Proof</u> (Kitaev Thm.)

<u>Hamiltonian</u> (not fully local yet!)

$$H = H_{in} + H_{prop} + H_{out}$$

Initially, take

$$\mathcal{H} = \underbrace{(\mathbb{C}^2)^{\otimes n}}_{\text{computation register}} \otimes \underbrace{\mathbb{C}^{T+1}}_{\text{clock register}} = \overbrace{\mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes |W|} \otimes (\mathbb{C}^2)^{\otimes |A|}}^{\text{computation register}} \otimes \mathbb{C}^{T+1}$$

output qubit; witness register qubits W; ancillas A

(will make clock local later).

$$H_{in} := \Pi_1^{(1)} \otimes |0\rangle\langle 0|_{cl} + \sum_{j \in A} \Pi_j^{(1)} \otimes |0\rangle\langle 0|_{cl}$$

forces output qubit & ancillas to "initially" be in $|0\rangle$ state

$$H_{prop} := \sum_{t=1}^{T-1} H_t, \quad \text{where}$$

$$H_t := \frac{1}{2} \mathbb{1} \otimes (|t\rangle\langle t| + |t+1\rangle\langle t+1|$$

forces clock to "tick"

$$- \frac{1}{2} U_{t+1} \otimes |t+1\rangle\langle t| - \frac{1}{2} U_{t+1}^\dagger \otimes |t\rangle\langle t+1|)$$

forward computation step — backwards step

$$H_{out} := \Pi_1^{(0)} \otimes |T\rangle\langle T|$$

gives energy "penalty" if output of QMA verifier computation is "0".

Lemma

$$H_{prop} \sim \mathbb{1} \otimes E$$

(i.e. $\exists$ unitary W s.t. $W H_{prop} W^\dagger = \mathbb{1} \otimes E$)

where $E = \sum_t \frac{1}{2} \left(|t\rangle - |t+1\rangle \right) \left(\langle t| - \langle t+1| \right)$

$$= \begin{pmatrix} 1/2 & -1/2 & & & & \\ -1/2 & 1 & -1/2 & & & \\ & -1/2 & 1 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & -1/2 \\ & & & & -1/2 & 1/2 \end{pmatrix}$$

Proof

Define unitary

$$W := \sum_{t=0}^{T} \prod_{i=1}^{t} U_i^\dagger \otimes |t\rangle\langle t| \qquad \left(\text{where } \prod_{i=1}^{0} U_i := \mathbb{1}\right)$$

"undoes" t steps of computation

$$= \mathbb{1} \otimes |0\rangle\langle 0| + \sum_{t=1}^{T} \left(U_1^\dagger \cdot U_2^\dagger \cdots U_{t-1}^\dagger \cdot U_t^\dagger \right) \otimes |t\rangle\langle t|$$

$W H_t W^\dagger$

$$= \left(U_1^\dagger \cdots U_t^\dagger \otimes |t\rangle\langle t| + U_1^\dagger \cdots U_t^\dagger \cdot U_{t+1}^\dagger \otimes |t+1\rangle\langle t+1| \right)$$

$$\cdot \frac{1}{2} \Big(\mathbb{1} \otimes \left(|t\rangle\langle t| + |t+1\rangle\langle t+1| \right)$$

$$- \frac{1}{2} U_{t+1} \otimes |t+1\rangle\langle t| - \frac{1}{2} U_{t+1}^\dagger \otimes |t\rangle\langle t+1| \Big)$$

$$\cdot \left(U_t \cdots U_1 \otimes |t\rangle\langle t| + U_{t+1} U_t \cdots U_1 \otimes |t+1\rangle\langle t+1| \right)$$

$$
\begin{aligned}
= \frac{1}{2} \Big(& \overbrace{(U_1^\dagger \cdots U_t^\dagger)(U_t \cdots U_1)}^{=\mathbb{1}} \otimes |t\rangle \overbrace{\langle t|t\rangle}^{=1} \\
& + \overbrace{(U_1^\dagger \cdots U_t^\dagger U_{t+1}^\dagger)(U_{t+1} U_t \cdots U_1)}^{=\mathbb{1}} \otimes |t+1\rangle\langle t+1| \\
& - \overbrace{(U_1^\dagger \cdots U_t^\dagger U_{t+1}^\dagger) U_{t+1} (U_t \cdots U_1)}^{=\mathbb{1}} \otimes |t+1\rangle\langle t| \\
& - \overbrace{(U_1^\dagger \cdots U_t^\dagger) U_{t+1}^\dagger (U_{t+1} U_t \cdots U_1)}^{=\mathbb{1}} \otimes |t\rangle\langle t+1| \Big)
\end{aligned}
$$

$$
= \mathbb{1} \otimes |\phi_t\rangle\langle\phi_t| \quad \text{where} \quad |\phi_t\rangle = \frac{1}{\sqrt{2}} \left(|t\rangle - |t+1\rangle \right)
$$

$$
H_{prop} \sim W H_{prop} W^\dagger = \sum_t \mathbb{1} \otimes |\phi_t\rangle\langle\phi_t| = \mathbb{1} \otimes E.
$$

$\square$

YES instance:

$\exists$ witness $|w\rangle$ s.t.

Pr (circuit outputs "0" on input $|w\rangle$)
$\leq \varepsilon$ (Def. QMA)

Let $$|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\psi_t\rangle |t\rangle_{cl}$$

where $$|\psi_t\rangle = U_t \cdot U_{t-1} \cdots U_2 \cdot U_1 \, |\psi_0\rangle$$

$$|\psi_0\rangle = \underset{\text{output qubit}}{|0\rangle} \; \underset{\text{witness}}{|w\rangle} \; \underbrace{|0\rangle \cdots |0\rangle}_{\text{ancillas}}$$

Note:

- $$|\Psi\rangle = \left(\sum_t U_t \cdots U_1 \otimes |t\rangle\langle t|\right)\left(|\psi_0\rangle \otimes \underbrace{\frac{1}{\sqrt{T+1}} \sum_t |t\rangle}_{|\varphi_0\rangle}\right)$$
  $$= W^\dagger |\psi_0\rangle \otimes |\varphi_0\rangle.$$
- Pr(output "0" on input $|w\rangle$)
  $$= \langle \psi_T | \pi_1^{(0)} | \psi_T\rangle \leq \varepsilon.$$

Want to show $\lambda_0(H) \leq \alpha$, α small
$\rightarrow$ show computational history state has low energy.

$$\lambda_0(H) = \min_{|\phi\rangle} \langle\phi| H |\phi\rangle$$

$$\leq \langle\Psi| H |\Psi\rangle$$

$$= \langle\Psi| H_{in} |\Psi\rangle + \langle\Psi| H_{prop} |\Psi\rangle + \langle\Psi| H_{out} |\Psi\rangle$$

$$= \sim\sim\sim \text{ " } \sim\sim\sim + \sum_t \langle\Psi| H_t |\Psi\rangle + \sim\sim\sim \text{ " } \sim\sim\sim$$

$$\langle\Psi| H_{in} |\Psi\rangle$$

$$= \frac{1}{T+1} \left(\sum_{t'=0}^{T} \langle\psi_{t'}| \langle t'| \right) \left(\left(\Pi_1^{(1)} + \sum_{j \in anc} \Pi_j^{(1)} \right) \otimes |0\rangle\langle 0|_{cL} \right) \cdot \left(\sum_{t=0}^{T} |\psi_t\rangle |t\rangle \right)$$

$$\propto \langle\psi_0| \langle 0| \left(\left(\Pi_1^{(1)} + \sum_{j \in anc} \Pi_j^{(1)} \right) \otimes |0\rangle\langle 0|_{cL} \right) |\psi_0\rangle |0\rangle$$

$$= \langle 0| \langle w| \langle 0|^{\otimes A} \cdot \Pi_1^{(1)} \cdot |0\rangle |w\rangle |0\rangle$$

$$= \langle 0| \overset{0}{\cancel{\Pi^{(1)}}} |0\rangle = 0.$$

$$\langle\Psi| H_{prop} |\Psi\rangle$$

$$= \langle\psi_0| \otimes \langle\varphi_0| W H_{prop} W^\dagger |\psi_0\rangle \otimes |\varphi_0\rangle$$

$$= \langle\psi_0| \otimes \langle\varphi_0| \mathbb{1} \otimes E |\psi_0\rangle \otimes |\varphi_0\rangle \qquad (W H_{prop} W^\dagger = \mathbb{1} \otimes E)$$

$$= \langle\varphi_0| E |\varphi_0\rangle$$

$$\propto \langle\varphi_0| \sum_t (|t\rangle - |t+1\rangle)(\langle t| - \langle t+1|) \sum_{t'} |t'\rangle$$

$$= 0.$$

$$\langle \Psi | H_{out} | \Psi \rangle$$

$$= \frac{1}{T+1} \left(\sum_{t'=0}^{T} \langle \psi_{t'} | \langle t' | \right) \left(\Pi_1^{(0)} \otimes |T\rangle\langle T| \right) \cdot$$

$$\cdot \left(\sum_{t=0}^{T} |\psi_t\rangle |t\rangle \right)$$

$$= \frac{1}{T+1} \langle \psi_T | \Pi_1^{(0)} | \psi_T \rangle$$

$$= \frac{1}{T+1} \Pr(\text{output "0" on input } |w\rangle)$$

(from above)

$$\leq \frac{\varepsilon}{T+1}$$

Thus

$$\lambda_0(H) \leq \cancel{\langle \Psi | H_{in} | \Psi \rangle}^{=0} + \sum_{t=1}^{T} \cancel{\langle \Psi | H_t | \Psi \rangle}^{=0} + \langle \Psi | H_{out} | \Psi \rangle$$

$$\leq \frac{\varepsilon}{T+1} \quad \text{YES instances .}$$

NO instance:

$\forall\ |w\rangle$:

Pr (circuit outputs "0" on input $|w\rangle$)

$= \langle \psi_T |\ \Pi_1^{(0)}\ | \psi_T \rangle$

$\geq 1 - \varepsilon$ (Def. QMA)

Need following:

Def (angle between subspaces)

Subspaces $\mathcal{L}_1, \mathcal{L}_2$.

$$\cos\theta\,(\mathcal{L}_1, \mathcal{L}_2) := \max_{\substack{|\varphi_1\rangle \in \mathcal{L}_1 \\ |\varphi_2\rangle \in \mathcal{L}_2}} |\langle \varphi_1 | \varphi_2 \rangle|$$

(cf. $\vec{u} \cdot \vec{v} = \cos\theta$, $|\vec{u}| = |\vec{v}| = 1$)

Lemma (Kitaev)

If $A, B \geq 0$ with $\ker A \cap \ker B = \emptyset$ and $\lambda_{min}(A|_{\mathrm{supp}A}), \lambda_{min}(B|_{\mathrm{supp}B}) \geq \mu$, then

$$A + B \geq 2\mu \sin^2 \frac{\theta}{2}$$

where $\theta = \theta(\ker A, \ker B)$.

(Recall $A \geq c \Leftrightarrow A - c\mathbb{1} \geq 0 \Leftrightarrow \lambda_{min}(A) \geq c$.)

Proof

Let Π_A = projector onto $\ker A$

Π_B = —— " —— $\ker B$

$A \geq \mu\,(\mathbb{1} - \Pi_A)$, $\quad B \geq \mu\,(\mathbb{1} - \Pi_A)$ (trivial)

$\rightarrow$ sufficient to prove

$$(\mathbb{1} - \Pi_A) + (\mathbb{1} - \Pi_B) \geq 2 \sin^2 \frac{\theta}{2}$$

or

$$\Pi_A + \Pi_B \leq \overbrace{1 + \cos\theta}^{= 2\cos^2 \frac{\theta}{2}} . \qquad (*)$$

Let $|\psi\rangle$ be any eigvect. of $\Pi_A + \Pi_B$ with eigval λ (> 0 since $\ker A \cap \ker B = \emptyset$):

$$\lambda |\psi\rangle = (\Pi_A + \Pi_B)|\psi\rangle = a\,|\varphi_A\rangle + b\,|\varphi_B\rangle$$

where $\Pi_A |\psi\rangle =: a\,|\varphi_A\rangle$, $\quad \Pi_B|\psi\rangle =: b\,|\varphi_B\rangle$

& wlog $a, b \in \mathbb{R}^+$ (absorb phases into $|\varphi_{A,B}\rangle$)

$$\lambda = \langle\psi|\,(\Pi_A + \Pi_B)\,|\psi\rangle$$

$$= \langle\psi|\,\Pi_A\,\Pi_A\,|\psi\rangle + \langle\psi|\,\Pi_B\,\Pi_B\,|\psi\rangle$$

($\Pi^2 = \Pi$ for projectors)

$$= a^2 + b^2$$

$$\lambda^2 = \langle\psi| \, (\Pi_A + \Pi_B)^2 \, |\psi\rangle$$

$$= \underbrace{a^2 + b^2}_{=\lambda} + 2ab \, \mathrm{Re}\langle\varphi_A|\varphi_B\rangle$$

$$\leq \lambda + 2ab \, |\mathrm{Re}\langle\varphi_A|\varphi_B\rangle| \qquad a, b \in \mathbb{R}^+$$

$$\leq \lambda + (a^2 + b^2) \, |\mathrm{Re}\langle\varphi_A|\varphi_B\rangle|$$

$$(a^2 + b^2 - 2ab = (a-b)^2 \geq 0 \Rightarrow 2ab \leq a^2 + b^2)$$

$$= \lambda \, (1 + |\mathrm{Re}\langle\varphi_A|\varphi_B\rangle|)$$

$$\leq \lambda \left(1 + \max_{\substack{|\varphi_A\rangle \in \ker A \\ |\varphi_B\rangle \in \ker B}} |\langle\varphi_A|\varphi_B\rangle|\right)$$

$$= \lambda \, (1 + \cos\theta)$$

$\therefore \; \lambda \leq 1 + \cos\theta$ which proves (*). $\square$

Let $A = H_{in} + H_{out}$

$B = H_{prop} = \sum_t H_t$

in Lemma.

Need to bound $\lambda_{min}(A|_{supp A})$, $\lambda_{min}(B|_{supp B})$, $\Theta(\ker A, \ker B)$

smallest non-zero eigval

Claim $\lambda_{min}(A|_{supp A}) \geq 1$

Proof: H_{in}, H_{out} commuting projectors

Claim $\lambda_{min}(B|_{supp B}) \geq \Omega(T^{-2})$

Proof

Define unitary

$$W := \sum_t (U_1^\dagger \cdot U_2^\dagger \cdots U_{t-1}^\dagger \cdot U_t^\dagger) \otimes |t\rangle\langle t|$$

(We used this before in proof of $H \geq 0$.)

$$B = H_{prop} \sim W H_{prop} W^\dagger = \mathbb{1} \otimes E$$

where $E = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 1 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & 1 & -\frac{1}{2} \\ & & & & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ from before

$\therefore$ Eigvals of B = eigvals of E.

Eigvals & eigvects of E are:

$$\lambda_k = 1 - \cos q_k \quad , \qquad |\varphi_k\rangle \propto \sum_{j=0}^{T} \cos\left(q_k \left(j + \tfrac{1}{2}\right)\right) |j\rangle$$

where

$$q_k = \frac{\pi k}{T+1} \quad , \quad k = 0, \ldots, T$$

<u>Exercise</u> Verify this.

$$\lambda_0 = 0, \quad |\varphi_0\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle .$$

Smallest non-zero eigval

$$\lambda_1 = 1 - \cos\left(\frac{\pi}{T+1}\right) \geq \Omega(T^{-2}). \quad \square$$

<u>Claim</u> $\sin^2\frac{\theta}{2} = \Omega\left(\frac{1-\sqrt{\varepsilon}}{T+1}\right)$, $\quad \theta \equiv \theta(\ker A, \ker B)$

<u>Proof</u>

$$A = H_{in} + H_{out}$$

$$= \left(\Pi_1^{(1)} + \sum_{j\in A} \Pi_j^{(1)}\right) \otimes |0\rangle\langle 0|_{cl} + \Pi_1^{(0)} \otimes |T\rangle\langle T|$$

$$\ker A = \operatorname{span}\Big\{ |0\rangle_1 |\psi\rangle \overbrace{|0\rangle\cdots|0\rangle}^{\text{ancillas}} \otimes |0\rangle_{cl},$$
$$|\varphi\rangle \otimes |t\rangle_{cl} \quad 0 < t < T,$$
$$|1\rangle_1 |\phi\rangle \otimes |T\rangle_{cl} \Big\}$$

$$\therefore \Pi_A = |0\rangle\langle 0|_1 \otimes \mathbb{1} \otimes (|0\rangle\cdots|0\rangle)(\langle 0|\cdots\langle 0|) \otimes |0\rangle\langle 0|_{cl}$$
$$+ \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle\langle t| + |1\rangle\langle 1|_1 \otimes |T\rangle\langle T|_{cl}$$

$$W \Pi_A W^\dagger = \overbrace{|0\rangle\langle 0|_1 \otimes \mathbb{1} \otimes (|0\rangle\cdots|0\rangle)(\langle 0|\cdots\langle 0|) \otimes |0\rangle\langle 0|_{cl}}^{\Pi_I}$$
$$+ \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle\langle t|_{cl}$$
$$+ \underbrace{U^\dagger\left(|1\rangle\langle 1|_1 \otimes \mathbb{1}_{[n]\setminus 1}\right) U}_{\Pi_F} \otimes |T\rangle\langle T|_{cl}$$

where $U := U_T U_{T-1} \cdots U_2 U_1$.

$$\cos^2\Theta = \max_{\substack{|\xi\rangle \in \ker A \\ |\eta\rangle \in \ker B}} |\langle \eta | \xi \rangle|^2 \qquad \text{by Def.}$$

$$= \max_{|\xi\rangle, |\eta\rangle} |\langle \eta | \Pi_A | \xi \rangle|^2 \qquad \Pi_A = \text{proj. on } \ker A$$

$$= \max_{|\eta\rangle \in \ker B} \langle \eta | \Pi_A | \eta \rangle \qquad \text{Cauchy-Schwarz}$$

$$\left(\text{saturate } |\langle \eta | \Pi_A | \xi \rangle|^2 \leq \| \Pi_A |\eta\rangle \| \cdot \| |\xi\rangle \| = \langle \eta | \Pi_A | \eta \rangle\right)$$

$$= \max_{|\eta\rangle \in \ker B} \langle \eta | W^\dagger (W \Pi_A W^\dagger) W | \eta \rangle$$

$$= \max_{|\eta'\rangle \in \ker WBW^\dagger} \langle \eta' | \Big(\Pi_I \otimes |0\rangle\langle 0|_{cl} + \Pi_F \otimes |T\rangle\langle T|_{cl} + \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle\langle t|_{cl} \Big) |\eta'\rangle$$

$$= \frac{1}{T+1} \max_{|\varphi\rangle} \sum_{t=0}^{T} \langle \varphi | \langle t | \left(\text{———} \right) \sum_{t'=0}^{T} |\varphi\rangle |t'\rangle$$

$$WBW^\dagger = \mathbb{1} \otimes E \Rightarrow |\eta\rangle = |\varphi\rangle \otimes \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle$$

$$= \frac{1}{T+1} \max_{|\varphi\rangle} \Big(\langle \varphi | (\Pi_I + \Pi_F) | \varphi \rangle + T - 1 \Big)$$

$$= \frac{1}{T+1} \max_{|\varphi\rangle} \langle \varphi | \Pi_I + \Pi_F | \varphi \rangle + \frac{T-1}{T+1}$$

$$\leq \frac{1 + \cos\vartheta}{T+1} + \frac{T-1}{T+1} \qquad \text{by Kitaev Lemma}$$

where $\vartheta = \Theta(\operatorname{supp} \Pi_I, \operatorname{supp} \Pi_F)$.

$$\cos^2\vartheta = \max_{\substack{|\eta\rangle \in \operatorname{supp}\Pi_I \\ |\xi\rangle \in \operatorname{supp}\Pi_F}} |\langle \eta | \xi \rangle|^2$$

$$= \max_{|w\rangle, |\varphi\rangle} \Big| \underbrace{\left({}_1\langle 0| \langle w| \langle 0| \cdots \langle 0|\right)}_{= |\Psi_0\rangle} \cdot \left(U^\dagger |1\rangle_1 |\varphi\rangle \right) \Big|^2$$

$$\Pi_I = |0\rangle_1\langle 0| \otimes \mathbb{1} \otimes (|0\rangle \cdots |0\rangle)(\langle 0| \cdots \langle 0|)$$

$$\Rightarrow |\eta\rangle = |0\rangle_1 |w\rangle |0\rangle \cdots |0\rangle$$

$$\Pi_F = U^\dagger (|1\rangle_1\langle 1| \otimes \mathbb{1}) U$$

$$\Rightarrow |\xi\rangle = |1\rangle_1 |\varphi\rangle$$

$$= \max_{|w\rangle, |\varphi\rangle} (\langle \Psi_0 | U^\dagger) \, |1\rangle_1\langle 1| \otimes |\varphi\rangle\langle\varphi| \, (U |\Psi_0\rangle)$$

$$\leq \max_{|w\rangle} \langle \Psi_T | \, \Pi_1^{(1)} \, |\Psi_T\rangle$$

$$\leq \varepsilon. \qquad \text{NO instance !}$$

$$\therefore \cos^2\theta \leq \frac{1+\cos\vartheta}{T+1} + \frac{T-1}{T+1} \leq 1 - \frac{1-\sqrt{\varepsilon}}{T+1}$$

$$\sin^2\frac{\theta}{2} \geq \frac{1-\cos^2\theta}{8\cos^2\theta} \geq \frac{1-\sqrt{\varepsilon}}{8(T+1)}. \qquad \square$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\& \ \cos^2\frac{\theta}{2} \leq 2\cos^2\theta \ \text{ for } \ 0 \leq \theta \leq \frac{\pi}{4}$$

Summary:

$$\lambda_{min}(A|_{supp A}) \geq 1$$

$$\lambda_{min}(B|_{supp B}) \geq \Omega(T^{-2})$$

$$\sin^2\frac{\theta}{2} \geq \frac{1-\sqrt{\varepsilon}}{8(T-1)}$$

Kitaev's Lemma $\Rightarrow$ for NO instance have

$$H = A + B \geq \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right)$$

i.e. have lower bound on λ_{min}:

$$\lambda_{min}(H) = \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right)$$ as required.

We have proven:

YES $\Rightarrow$ $\lambda_{min}(H) \leq \frac{\varepsilon}{T+1} =: \alpha$

NO $\Rightarrow$ $\lambda_{min}(H) \geq \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right) =: \beta$

Taking $\varepsilon = \Omega\left(\frac{1}{T^3}\right) \Rightarrow \beta - \alpha = \Omega\left(\frac{1}{T^3}\right)$

$\therefore$ Have proven reduction from QMA to Local Hamiltonian ...

... except that clock register has dimension $T+1$

$\longrightarrow$ H not on qudits with const. local dim. d.

Could split clock into $\log T$ qubits

= <u>binary</u> clock $\quad (\mathbb{C}^T \cong (\mathbb{C}^2)^{\otimes \log T})$

$\longrightarrow$ local terms $\log T$-local, not k-local (k const).

(Example of general phenomenon: can always trade off locality against local dimension.)