I. Hamiltonian Complexity

Notation & Terminology

- $\lambda_0(H)$ or $\lambda_{min}(H) := \min_{|\phi\rangle} \langle\phi| H |\phi\rangle$
  "ground state energy"

- $H|\psi_0\rangle = \lambda_0|\psi_0\rangle$ $\quad |\psi_0\rangle$ = "ground state"

- $\mathcal{L}_0 = \text{span}\{|\psi\rangle : H|\psi\rangle = \lambda_{min}|\psi\rangle\}$
  $\mathcal{L}_0$ = "g.s. subspace"

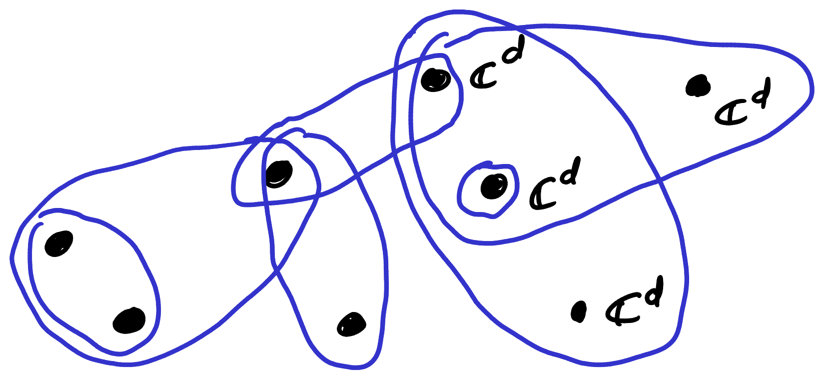
- Note: $H \geq 0$, $\lambda_{min}(H) = 0 \Rightarrow$ g.s. subspace $= \ker H$

- $\Delta(H) = $ "$\lambda_1 - \lambda_0$" $= \min_{|\phi\rangle \perp \mathcal{L}_0} \langle\phi|H|\phi\rangle - \lambda_0(H)$
  "spectal gap": <u>same</u> operator, <u>different</u> eigvals

- $\lambda_0(H_1) - \lambda_0(H_2)$
  "promise gap": <u>different</u> operators, <u>same</u> eigvals

Local Hamiltonians

Core object of study for this entire course.



Many-body quantum system:

- Multipartite Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$
  (often $d=2$: n qubit system)
  e.g. particles with spin-s $\left(d = \frac{s(s+1)}{2}\right)$

- Local interaction Hamiltonian:
  $$h = h_S \otimes \mathbb{1}_{[n]\setminus S} \quad , \quad S \subset [n]$$
  i.e. h acts non-trivially only on subset S of the particles

  We say that interaction h is <u>k-local</u> if $|S| = k$ (i.e. acts non-trivially on k particles).

<u>Notation</u>: often write "h_S" $\equiv h_S \otimes \mathbb{1}_{[n]\setminus S}$, i.e. subscript indicates indices of particles acted on, $\mathbb{1}$ on rest is implicit.

Def (k-local Hamiltonian)

$$H = \sum_i h^{(i)} \in \mathcal{B}((\mathbb{C}^d)^{\otimes n})$$

We say that H is a k-local Hamiltonian (or "H is k-local") if $\forall i \;\; h^{(i)}$ is k-local.

I.e. k-local many-body Hamiltonian made of many interactions, each involving at most k particles.

Example: spins on line with nearest-neighbour interactions: $H = \sum_{i=1}^{n-1} h_{i,i+1}$

spins on lattice —— " —— —— " —— : $H = \sum_{\langle i,j \rangle} h_{ij}$

neighbours

Note: In general, no underlying geometry
$\Rightarrow$ no requirement that local interactions are geometrically local.

Physics terminology: "k-local" = "k-body" but stuck with "k-local" now in QIT.

Motivation:

Fundamental interaction n nature (e.g. electromagnetic, or "Coulomb", interaction) are not k-local — interactions decay as $\frac{1}{\text{poly}}$, not finite-range.

However, in many-body system electron clouds of neighbouring atoms shield atoms from fields of more distant particles (cf. Faraday cage).

At low-energies, many-body systems often well-approximated by k-local (often 2-local) Hamiltonians.
$\longrightarrow$ k-local H crop up throughout condensed matter physics.

The major topic of cond. mat.: phase transitions.

Quantum phase transitions (e.g. superconductivity, superfluidity) occur at zero or very low temperature characterised by abrupt change in ground state of system

$\longrightarrow$ quantum condensed matter $\approx$ study of ground states of many-body systems.

The "Local Hamiltonian problem" asks whether or not a local Hamiltonian has a low energy ground state.

Def (Local Hamiltonian Problem)

Input: k-local Hamiltonian H on n qudits with m local terms,

"Promise" $\begin{cases} \text{where } \lambda_0(H) \lesssim \alpha \text{ or } \lambda_0(H) \gtrsim \beta \\ \text{with } \beta - \alpha \geq \frac{1}{\text{poly}(n)} \end{cases}$

Output: YES if $\lambda_0(H) \leq \alpha$

NO if $\lambda_0(H) \geq \beta$

Note that:

- Input is classical description of H, e.g. specify d^{2k} matrix elements for each of the m terms.
- Wlog $m \leq \binom{n}{k} = O(n^k) = \text{poly}(n)$

  problem size (for fixed k, d) scales as $\text{poly}(n)$

  $\longrightarrow$ take n to measure problem size

- (Strictly speaking, number of bits of data required to specify input also depends on precision of matrix entries. All our results will hold if matrix entries are restricted to poly(n) digits of precision, so we ignore the input precision from now on.)

- Together with input, given promise that condition
$\lambda_0(H) \leq \alpha$ or $\lambda_0(H) \geq \beta$, $\quad \beta - \alpha = \frac{1}{\mathrm{poly}(n)}$
holds for the input

  $\longrightarrow$ only required to solve problem under assumption condition holds.

- Conversely, to prove hardness results must <u>show</u> condition <u>does</u> hold for any H we construct.

Thm (Kitaev)
The Local Hamiltonian problem is QMA-hard

The first & most important result in Hamiltonian Complexity theory.

Has many interesting implications for QIT, CompSci, & especially physics (see later)

To prove QMA-hardness:
Transform any QMA prob. $\longrightarrow$ Local H. prob.

Recall Def. QMA:
$\exists$ poly-sized q. circuit $U = U_T \cdot U_{T-1} \cdots U_2 \cdot U_1$ s.t.
YES: $\exists\ |w\rangle$: $\Pr(U|w\rangle$ outputs "1"$) \geq 2/3$
NO: $\forall\ |w\rangle$: $\Pr(U|w\rangle$ " " $) \leq 1/3$.

Idea:
"Encode" verifier circuit $\longrightarrow$ local Hamiltonian s.t. ground state encodes output of circuit.

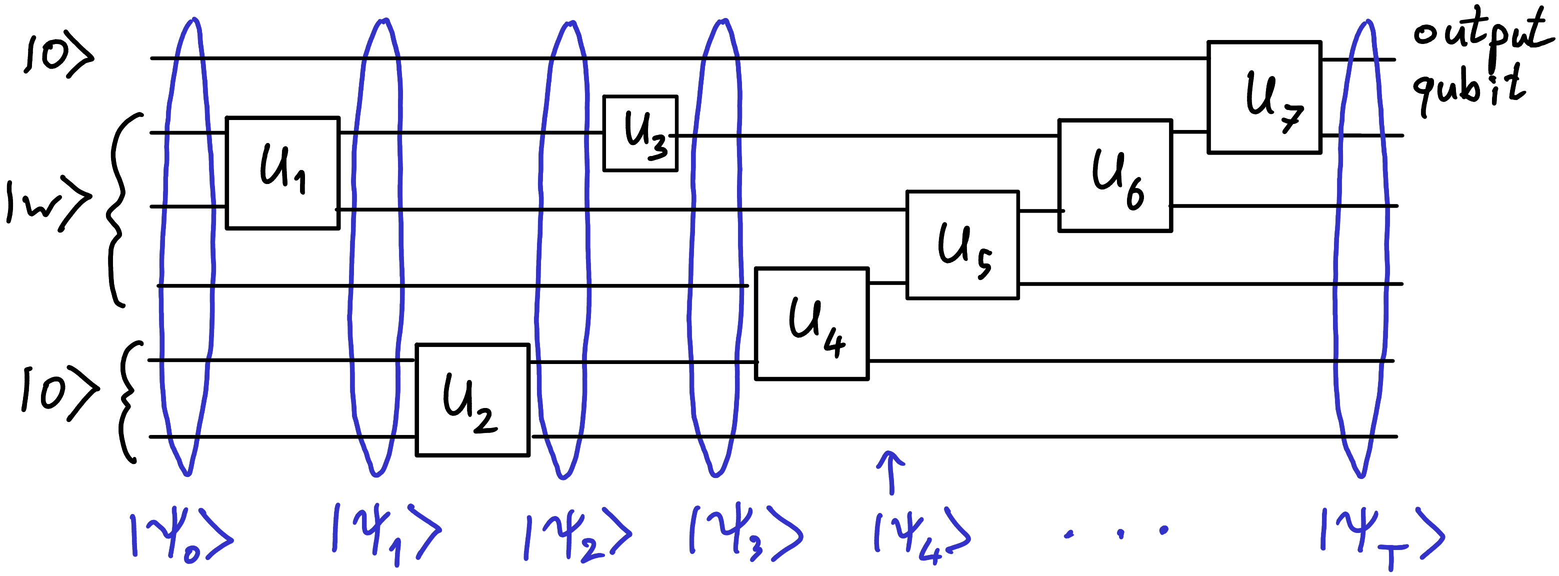
Add term that gives additional energy "penalty" if circuit outputs "0".

YES / NO $\longleftrightarrow$ low-energy / high energy g.s.

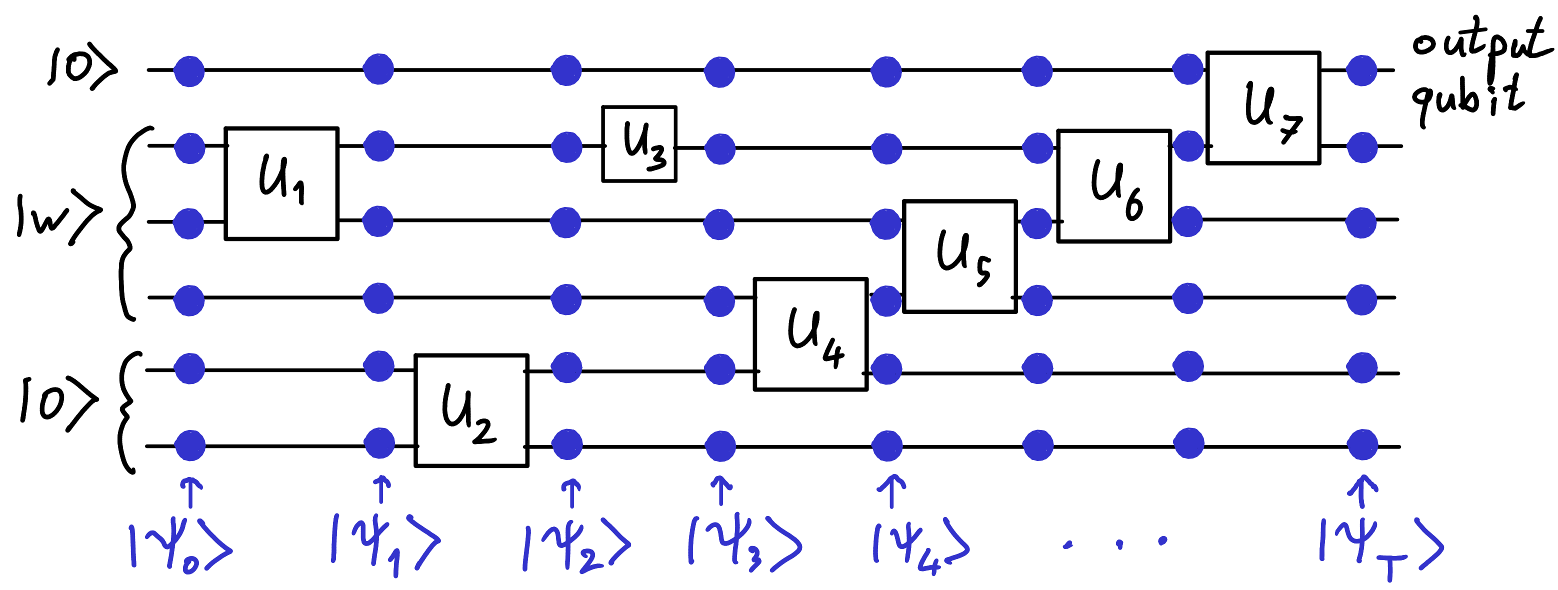
Before proving Thm, we first consider a naive approach that doesn't work, but teaches us something important about quantum many-body states and Hamiltonians.

Naive proof approach (doesn't work!)

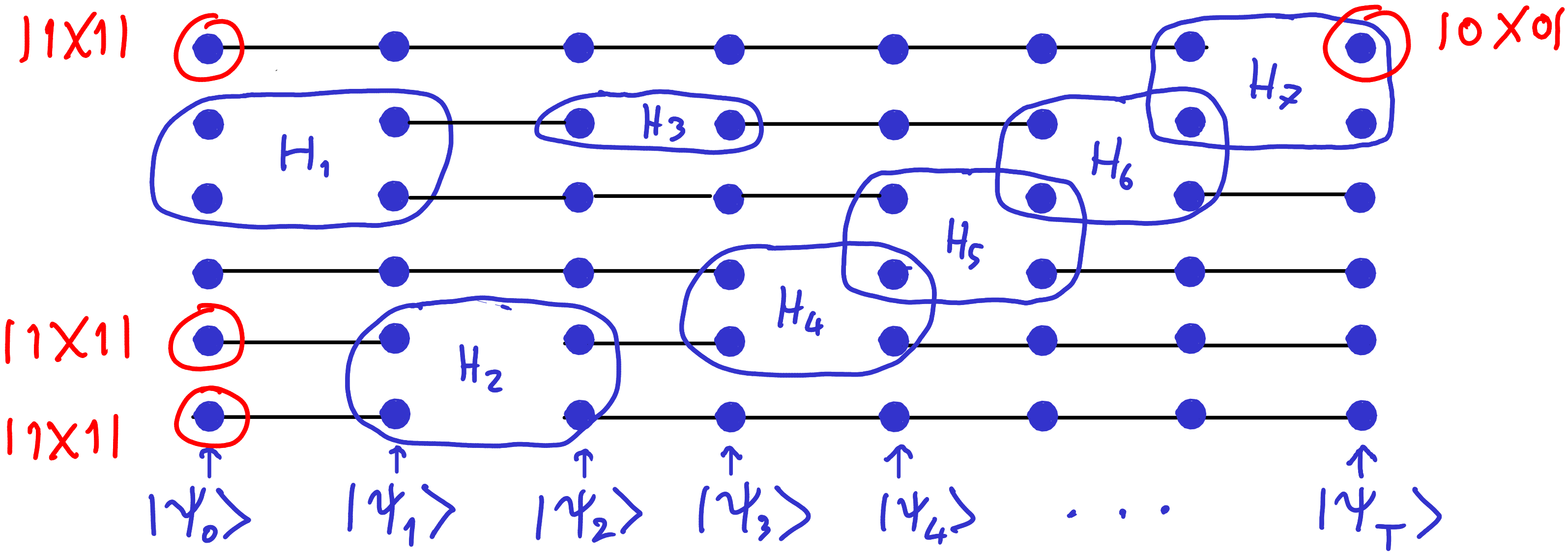
Verifier circuit:



Imagine laying out circuit on lattice of qubits:



Now put local Hamiltonians wherever there's a gate (also need 2-local Hamiltonian terms for identity gates):



Can easily force qubits representing initial state of output & ancilla qubits to be $|0\rangle$ using 1-local term $\Pi^{(1)} = |1\rangle\langle 1|$,

But leave initial witness qubits state unconstrained

Similarly, can give additional energy penalty if qubit representing <u>final</u> state of output qubit is "0" using 1-local term $\Pi^{(0)}_{out} = |0\rangle\langle 0|$

If we could find Hamiltonian terms $H_1, \ldots, H_T$ s.t. ground states represent correct evolution of circuit, we'd be done:

YES instance:

$\exists$ witness $|w\rangle$ s.t. $\Pr(\text{output } |1\rangle) \geq \frac{2}{3}$

$\Rightarrow \exists$ state $|\psi_0\rangle|\psi_1\rangle \cdots |\psi_T\rangle$ representing correct evolution of circuit s.t. only picks up energy $\leq \frac{1}{3}$ from $\Pi^{(0)}_{out}$

$\longrightarrow$ g.s. energy $\leq \frac{1}{3}$

NO instance:

$\forall$ $|w\rangle$, $\Pr(\text{output } |1\rangle) \leq \frac{1}{3}$

$\Rightarrow \forall$ $|\psi\rangle = |\psi_0\rangle|\psi_1\rangle \cdots |\psi_T\rangle$ representing correct evol$^{\underline{n}}$ of circuit, pick up energy $\geq \frac{2}{3}$ from $\Pi^{(0)}_{out}$

$\forall$ $|\psi\rangle$ not representing correct evol$^{\underline{n}}$, pick up large energy penalty from $H_1, \ldots, H_T$

$\longrightarrow$ g.s. energy $\geq \frac{2}{3}$

Exercise

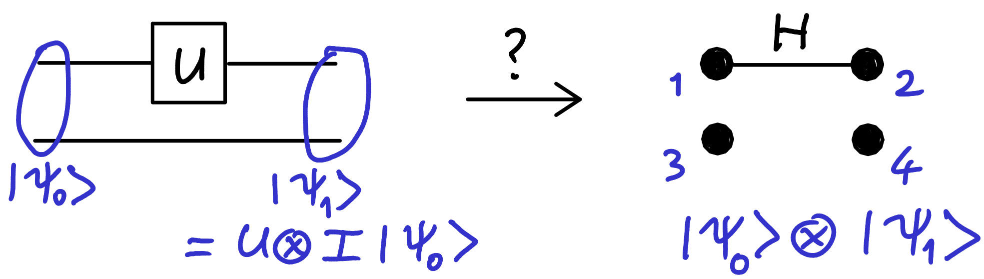
Make above argument rigorous.

If we could find Hamiltonian terms $H_1, \dots, H_T$ s.t. ground states represent correct evolution of circuit, we'd be done.

Exercise

Use this approach to prove NP-hardness of Local Hamiltonian problem for classical Hamiltonians (diagonal in computational basis).

Unfortunately, such Hamiltonian terms cannot exist in quantum case.



Claim

$\exists U : \nexists H_{12}$ s.t. ground state subspace

$$\mathcal{L}_0(H_{12} \otimes \mathbb{1}_{34}) = \text{span}\{ |\psi\rangle_{13} \otimes (U \otimes \mathbb{1}) |\psi\rangle_{24} \}.$$

Proof

Wlog can take $H \geq 0$, $\lambda_{min}(H) = 0$ (substitute $H' = H + \lambda_{min}(H) \cdot I$, eigenvectors unchanged).

$\longrightarrow \mathcal{L}_0(H \otimes \mathbb{1}) = \ker(H \otimes \mathbb{1})$.

Consider 2-qubit Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ with trivial gate $U = \mathbb{1}$, so

$$\mathrm{span}\{|\psi\rangle_{13} \otimes (U \otimes \mathbb{1})|\psi\rangle_{24}\} = \mathrm{span}\{|\psi\rangle_{13}|\psi\rangle_{24}\}.$$

Let $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. entangled state! (ebit)

$|\phi\rangle|\phi\rangle \in A_0$, so

$$\begin{aligned}
0 &= {}_{13}\langle\phi|\,{}_{24}\langle\phi|\,(H_{12} \otimes \mathbb{1}_{34})\,|\phi\rangle_{13}|\phi\rangle_{24} \\
&= \mathrm{tr}\left[H_{12} \otimes \mathbb{1}_{34} \cdot |\phi\rangle_{13}|\phi\rangle_{24}\langle\phi|_{13}\langle\phi|_{24}\right] \\
&= \mathrm{tr}\left[H_{12} \cdot \mathrm{tr}_3(|\phi\rangle\langle\phi|_{13}) \otimes \mathrm{tr}_4(|\phi\rangle\langle\phi|_{24})\right] \\
&= \mathrm{tr}\left[H_{12} \cdot \mathbb{1}_1 \otimes \mathbb{1}_2\right] \\
&= \mathrm{tr}\, H.
\end{aligned}$$

$H \geq 0$, $\mathrm{tr}\, H = 0 \Rightarrow H = 0$.

$$\begin{aligned}
\mathcal{L}_0(H \otimes \mathbb{1}) &= \ker(0 \otimes \mathbb{1}) = \ker(0) \\
&= \mathcal{H} \neq \mathrm{span}\{|\psi\rangle|\psi\rangle\} \quad \square
\end{aligned}$$

<u>Exercise</u>

Show Proposition holds $\forall\, U$.