Example Sheet 2

A. Kitaev's Theorem

1. Consider the following Hamiltonian acting on a Hilbert space $\mathcal{H} \otimes \mathbb{C}^{T+1}$

$$H = \sum_{t=0}^{T} H_t \otimes |t\rangle\langle t| + \sum_t A_t \otimes |t\rangle\langle t+1| + A_t^\dagger \otimes |t+1\rangle\langle t|\,,$$

and the Hamiltonian H' on $\mathcal{H} \otimes (\mathbb{C}^2)^{\otimes T}$ formed by replacing terms in H as follows:

$$\begin{aligned}
|t\rangle\langle t| &\longmapsto |1\rangle\langle 1|_t \otimes |0\rangle\langle 0|_{t+1}\,, && 0 < t < T\\
|0\rangle\langle 0| &\longmapsto |0\rangle\langle 0|_1\\
|T\rangle\langle T| &\longmapsto |1\rangle\langle 1|_T\\
|t\rangle\langle t-1| &\longmapsto |1\rangle\langle 1|_{t-1} \otimes |1\rangle\langle 0|_t \otimes |0\rangle\langle 0|_{t+1}\,, && 0 < t < T\\
|1\rangle\langle 0| &\longmapsto |1\rangle\langle 0|_1 \otimes |0\rangle\langle 0|_2\\
|T\rangle\langle T-1| &\longmapsto |1\rangle\langle 1|_{T-1} \otimes |1\rangle\langle 0|_T
\end{aligned}$$

Let $\iota : \mathbb{C}^{T+1} \hookrightarrow (\mathbb{C}^2)^{\otimes T}$ be the embedding defined by $\iota\,|t\rangle = |1\rangle^{\otimes t}\,|0\rangle^{\otimes T-t}$, and define $\mathcal{L} = \iota\mathbb{C}^{T+1} = \mathrm{span}\{|1\rangle^{\otimes t}\,|0\rangle^{\otimes T-t}\}_{t=1\dots T}$.

Prove that the actions of H' on the subspace $\mathcal{H} \otimes \mathcal{L}$ is equivalent to the action of H on the original Hilbert space, i.e. $\iota H\,|\psi\rangle = H'\iota\,|\psi\rangle$.

What goes wrong if we instead use the replacement

$$|t\rangle\langle t-1| \longmapsto |1\rangle\langle 1|_{t-1} \otimes |1\rangle\langle 0|_t \qquad 0 < t < T\,?$$

2. Prove that the Local Hamiltonian problem is in QMA.

B. Norms and correlation functions

3. We can extend the definition of connected correlation functions to mixed states ρ in the obvious way: $C(X,Y) := \mathrm{Tr}(\rho XY) - \mathrm{Tr}(\rho X)\,\mathrm{Tr}(\rho Y)$. Show that this reduces to the standard definition for pure states.

   Prove that if a bipartite density matrix ρ_{AB} is close in trace-norm to a product state (i.e. $\|\rho - \sigma_A \otimes \sigma_B\|_1 \leq \epsilon$ for some density matrices σ_A, σ_B), then its connected correlation functions are small.

   Show that this is *not* necessarily true if ρ is close in trace-norm to a separable state (i.e. $\|\rho - \sum_i p_i \sigma_A^{(i)} \otimes \sigma_B^{(i)}\|_1 \leq \epsilon$). Why is this?

4. (Helstrom bound)
Given an ensemble $\mathcal{E} = \{(\rho_1, p_1), (\rho_2, p_2)\}$ (i.e. with probability p_1 we are given the state ρ_1, with probability p_2 we are given ρ_2, where $p_1 + p_2 = 1$), prove that the minimal probability or error when trying to distinguish states drawn from $\mathcal{E}$ is given by
$$P_E(\mathcal{E}) = \frac{1}{2}\left(1 - \|p_1\rho_1 - p_2\rho_2\|_1\right).$$
Furthermore, prove that the optimal measurement is projective.

C. Lieb-Robinson bounds

5. Let $f(t) = [A(t), B]$ satisfy the following inhomogeneous ordinary differential equation:
$$\frac{\mathrm{d}f}{\mathrm{d}t} = i[H, f(t)] + \big[\, i[H, A(t)],\, B\, \big] \tag{1}$$
Using the method of variation of parameters, or otherwise, solve this ODE to obtain the solution
$$f(t) = e^{iHt}[A(0), B]e^{-iHt} + \int_0^t e^{iH(t-s)} \big[\, i[H, A(s)],\, B\, \big]\, e^{-iH(t-s)}\, \mathrm{d}s.$$
Verify your solution by substituting back into Eq. (1).
Hint: it may help to note that commutation is a linear operation on operators.

6. (Lieb-Robinson bound for quasi-local interactions)
Let $d(i, j)$ be a metric on the sites i, j of a many body system, and let $H = \sum_Z h_z$ be a many-body Hamiltonian satisfying the following bound for some finite μ, s and for all sites i:
$$\sum_{Z \ni i} \|h_Z\| |Z| e^{\mu\, \mathrm{diam}(Z)} \le s,$$
where the diameter of a subset $\mathrm{diam}(Z) := \max_{i,j \in Z} d(i, j)$.
Prove operators A_X, B_Y acting non-trivially on disjoint subsets X and Y satisfy the following Lieb-Robinson bound:
$$\|[A(t), B]\| \le 2\|A_X\|\|B_Y\| \min(|X|, |Y|)\, e^{-\mu\, d(X,Y)} \left(e^{2st} - 1\right)$$
where A evolves under H (in the Heisenberg picture), and the distance between two subsets $d(X, Y) := \min_{i \in X, j \in Y} d(i, j)$.
Hint: recall that, by definition, any metric satisfies the triangle inequality $d(i, j) \le d(i, k) + d(k, j)$.