# Lieb-Robinson bounds

In relativistic QM, causality implies that space-like separated observables commute, and continue to commute as they evolve, until their light-cones intersect.

In non-relativistic QM, this is <u>not</u> true.

Let
$$A_X \equiv A_X \otimes \mathbb{1}_{rest}$$
$$B_Y \equiv B_Y \otimes \mathbb{1}_{rest}$$
act non-trivially on disjoint subsets $X$, $Y$ respectively of a many-body system, so

$$[A_X, B_Y] = 0.$$

In general, $\forall\, t > 0$,

$$[A_X(t), B_Y(t)] \neq 0.$$

Does causality break down completely in non-relativistic QM?

No! Lieb-Robinson bounds show an approximate version of causality does still hold.

There exists a finite speed of propagation of information in a many-body system, & observables <u>approximately</u> commute until their "light cones" intersect.

(Surprisingly, this dynamical result is very useful in proving static properties ...)

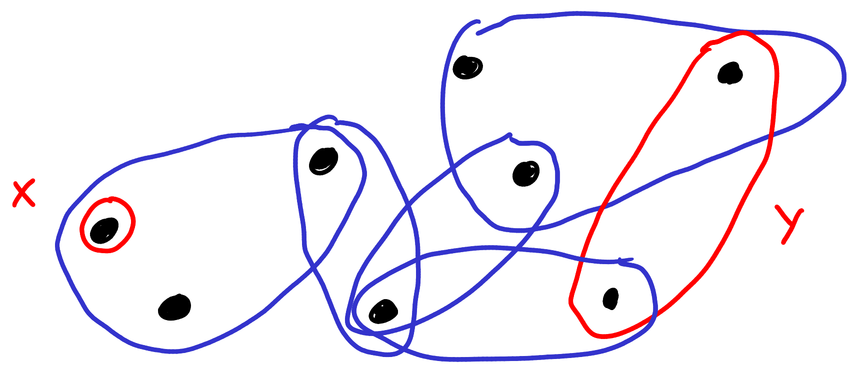
Def (interaction distance)

k-local Hamiltonian $H = \sum_Z h_Z$

subsets of qudits $X, Y$

$$d(X,Y) := \min \left| \left\{ Z_i : \begin{array}{l} X \cap Z_1 \neq \emptyset \\ Z_i \cap Z_{i+1} \neq \emptyset \\ Z_n \cap Y \neq \emptyset \end{array} \right\}_{i=1\ldots n} \right|$$

I.e. interaction distance is min # "hops" along interactions to get from $X$ to $Y$.




$d(X,Y) = 3$

**Theorem** (Lieb-Robinson)

k-local Hamiltonian $H = \sum_{Z} h_Z$ such that

$\exists\, \mu, s > 0$ s.t. $\forall$ qudits $i$ : $\sum_{Z \ni i} \| h_Z \| \le s e^{-\mu}$.

$A_X, B_Y$ operators on subsets of qudits $X, Y$.

Then

$$\| [A_X(t), B_Y] \|$$
$$\le 2 \|A_X\| \cdot \|B_Y\| \min(|X|, |Y|)\, e^{-\mu d(X,Y)} (e^{2kst} - 1).$$

**Proof**

Let $H = H_Y + H_{Y^c}$

where $H_Y = \sum_{Z \cap Y \neq \emptyset} h_Z$, $\quad H_{Y^c} = \sum_{Z \cap Y = \emptyset} h_Z$.

Note $[H_{Y^c}, B_Y] = 0$.

Write $f(t) = [A(t), B] \in \mathcal{B}(\mathcal{H})$.

$$\begin{aligned}
\frac{d}{dt} f(t) &= \frac{d}{dt}[A(t), B] \\
&= \frac{d}{dt} \left( e^{iHt} A e^{-iHt} B - B e^{iHt} A e^{-iHt} \right) \\
&= [\, i[H, A(t)], B] \\
&= i[H_{Y^c}, [A(t), B]] + [\, i[H_Y, A(t)], B] \\
&= i[H_{Y^c}, f(t)] + [\, i[H_Y, A(t)], B]
\end{aligned}$$

This is an inhomogeneous linear ODE for $f(t)$.

$\rightarrow$ Solve using variation of parameters
(Exercise) :

$$[A(t), B] = e^{iH_{Y^c}t}[A(0), B]\, e^{-iH_{Y^c}t} + \int_0^t e^{iH_{Y^c}(t-s)}[i[H_Y, A(s)], B]\, e^{-iH_{Y^c}(t-s)}\, ds$$

Taking norms :

$\| [A(t), B] \|$

$$\leq \| [A(0), B] \| + \int_0^t \| [i[H_Y, A(s)], B] \|\, ds$$

unitary invariance + triangle ineq.

$$\leq \| [A(0), B] \| + 2\|B\| \int_0^t \| [A(s), H_Y] \|\, ds$$

$\| [A, B] \| \leq 2\, \|A\| \cdot \|B\|$

$$= \| [A(0), B] \| + 2\|B\| \sum_{Z \cap Y \neq \emptyset} \int_0^t \| [A(s), h_Z] \|\, ds$$

Let $\quad C_A(Z,t) := \sup_{O_Z} \dfrac{\| [A_X(t), O_Z] \|}{\| O_Z \|} \geq 0$

Note:

- $\| [A(t), B] \| \leq \|B\| \, C_A(Y,t)$.
- $C_A(Z,0) \begin{cases} = 0 & X \cap Z = \emptyset \\ \leq 2\|A\| & X \cap Z \neq \emptyset \end{cases}$

$$= 2\|A\| \, \delta(X,Z)$$

$$\delta(X,Z) = \begin{cases} 0 & X \cap Z = \emptyset \\ 1 & X \cap Z \neq \emptyset \end{cases}$$

Rewriting, we have (from above):

$$\boxed{C_A(Y,t) \leq C_A(Y,0) + 2 \sum_{Z:\, Y \cap Z \neq \emptyset} \|h_Z\| \int_0^t C_A(Z,s)\, ds \qquad (*)}$$

$\left( f(x) \leq g(x) \;\Rightarrow\; \sup_x f(x) =: f(x^*) \leq g(x^*) \leq \sup_x g(x) \right)$

Iterating (*): (related to Picard iteration)

$C_A(Y,t)$

$$\leq 0 + 2\sum_{Z_1\cap Y\neq\emptyset}\|h_{Z_1}\|\int_0^t ds_1\, C_A(Z_1, s_1)$$

sum over $Z_1$ (drop "$Z_1$:" for brevity)

$$\leq 2\sum_{Z_1\cap Y\neq\emptyset}\|h_{Z_1}\|\int_0^t ds_1\Big(C_A(Z_1,0) + 2\sum_{Z_2\cap Z_1\neq\emptyset}\|h_{Z_2}\|\int_0^{s_1} ds_2\, C_A(Z_2,s_2)\Big)$$

using (*)

$$\leq 2\sum_{Z_1\cap Y\neq\emptyset}\|h_{Z_1}\|\int_0^t 2\|A\|\,\delta(X,Z_1) + 2^2\sum_{Z_1\cap Y\neq\emptyset}\|h_{Z_1}\|\sum_{Z_2\cap Z_1\neq\emptyset}\|h_{Z_2}\|\int_0^t ds_1\int_0^{s_1} ds_2\, C_A(Z_2,s_2)$$

using $C_A(Z,0)\leq 2\|A\|\,\delta(X,Z)$

$$\leq 2\|A\|\,(2t)\sum_{\substack{Z_1\cap Y\neq\emptyset\\ Z_1\cap X\neq\emptyset}}\|h_{Z_1}\| + 2^2\sum_{Z_1\cap Y\neq\emptyset}\|h_{Z_1}\|\sum_{Z_2\cap Z_1\neq\emptyset}\|h_{Z_2}\|\int_0^t ds_1\int_0^{s_1} ds_2\, 2\|A\|\,\delta(X,Z_2)$$

$$+ 2^3\sum_{Z_1\cap Y\neq\emptyset}\|h_{Z_1}\|\sum_{Z_2\cap Z_1\neq\emptyset}\|h_{Z_2}\|\sum_{Z_3\cap Z_2\neq\emptyset}\|h_{Z_3}\|\int_0^t ds_1\int_0^{s_1} ds_2\int_0^{s_2} ds_3\, C_A(Z_3,s_3)$$

using (*)

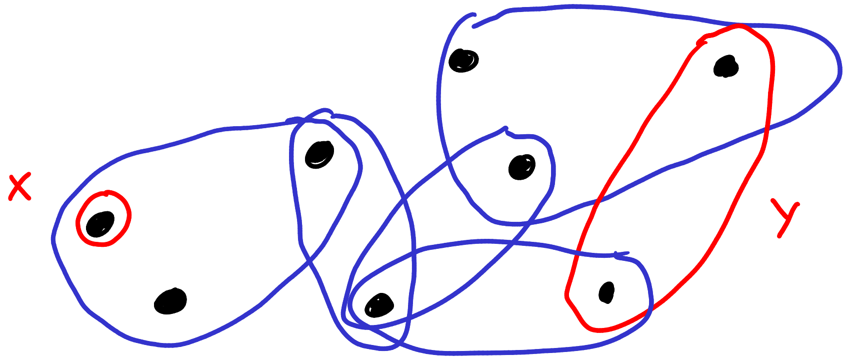
$$= 2\,\|A\|\,(2t) \sum_{\substack{Z_1 \cap Y \neq \phi \\ Z_1 \cap X \neq \phi}} \|h_{Z_1}\|$$

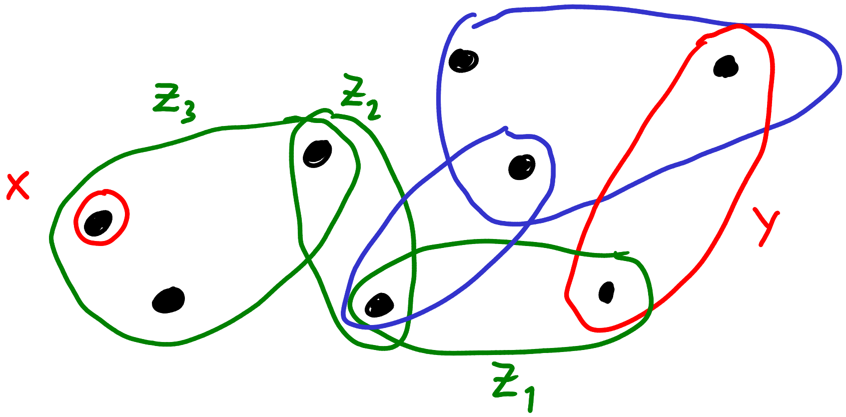
$$+ 2\,\|A\|\,\frac{(2t)^2}{2!} \sum_{Z_1 \cap Y \neq \phi} \|h_{Z_1}\| \sum_{\substack{Z_2 \cap Z_1 \neq \phi \\ Z_2 \cap X \neq \phi}} \|h_{Z_2}\|$$

$$+ 2\,\|A\|\,\frac{(2t)^3}{3!} \sum_{Z_1 \cap Y \neq \phi} \|h_{Z_1}\| \sum_{Z_2 \cap Z_1 \neq \phi} \|h_{Z_2}\| \sum_{\substack{Z_3 \cap Z_2 \neq \phi \\ Z_3 \cap X \neq \phi}} \|h_{Z_3}\|$$

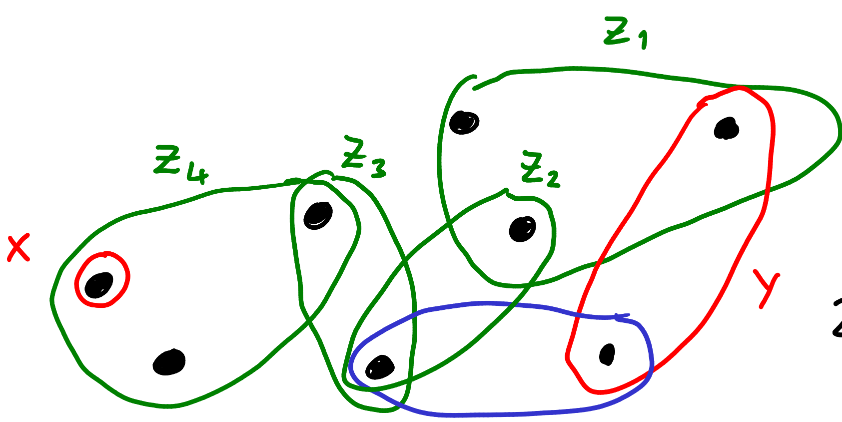
$$+ \cdots$$

$$= 2\,\|A\| \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} \sum_{Z_1 \cap Y \neq 0} \|h_{Z_1}\| \sum_{Z_2 \cap Z_1 \neq \phi} \|h_{Z_2}\| \cdots \sum_{\substack{Z_n \cap Z_{n-1} \neq \phi \\ Z_n \cap X \neq \phi}} \|h_{Z_n}\| . \qquad (**)$$




$$2\,\|A\|\,\frac{(2t)^3}{3!}\,\|h_{z_1}\|\,\|h_{z_2}\|\,\|h_{z_3}\|$$




$$2\,\|A\|\,\frac{(2t)^4}{4!}\,\|h_{z_1}\|\,\|h_{z_2}\|\,\|h_{z_3}\|\,\|h_{z_4}\|$$

Bound each term using assumption on interaction strengths in Thm:

$$\sum_{Z_1 \cap Y \neq 0} \| h_{Z_1} \| \cdots \sum_{Z_{n-1} \cap Z_{n-2} \neq \emptyset} \| h_{Z_{n-1}} \| \sum_{\substack{Z_n \cap Z_{n-1} \neq \emptyset \\ Z_n \cap X \neq \emptyset}} \| h_{Z_n} \|$$

$$\leq \sum_{i_1 \in Y} \sum_{Z_1 \ni i_1} \| h_{Z_1} \| \cdots \sum_{j \in Z_{n-2}} \sum_{Z_{n-1} \ni j} \| h_{Z_{n-1}} \| \sum_{k \in Z_{n-1}} \sum_{\substack{Z_n \ni k \\ Z_n \cap X \neq \emptyset}} \| h_{Z_n} \|$$

$$\leq \sum_{i \in Y} \sum_{Z_1 \ni i} \| h_{Z_1} \| \cdots \sum_{j \in Z_{n-2}} \sum_{Z_{n-1} \ni j} \| h_{Z_{n-1}} \| \sum_{k \in Z_{n-1}} s e^{-\mu}$$

$$\leq \sum_{i \in Y} \sum_{Z_1 \ni i} \| h_{Z_1} \| \cdots \sum_{j \in Z_{n-2}} \sum_{Z_{n-1} \ni j} \| h_{Z_{n-1}} \| \, k s e^{-\mu}$$

$$\leq \sum_{i \in Y} \sum_{Z_1 \ni i} \| h_{Z_1} \| \cdots \sum_{j \in Z_{n-2}} k s^2 e^{-2\mu}$$

$$\leq \sum_{i \in Y} \sum_{Z_1 \ni i} \| h_{Z_1} \| \cdots k^2 s^2 e^{-2\mu}$$

$$\leq \cdots$$

$$\leq \begin{cases} \sum_{i \in Y} (k s e^{-\mu})^n & \exists \text{ path } Y \to X \text{ of length } n \\ 0 & \nexists \text{ path} \end{cases}$$

$$\leq |Y| \, (ks)^n \, e^{-\mu d(X,Y)}$$

Inserting bound in (**), we have:

$$C_A(Y,t) \leq 2\|A\| \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} |Y| (ks)^n e^{-\mu d(X,Y)}$$
$$= 2\|A\| \, |Y| \, e^{-\mu d(X,Y)} \left(e^{2kst} - 1\right).$$

Note we could equally well have summed paths in (**) from other end, to get:

$$C_A(Y,t) \leq 2\|A\| \min(|X|,|Y|) \, e^{-\mu d(X,Y)} \left(e^{2kst} - 1\right).$$

Thm follows from $\|[A(t), B]\| \leq \|B\| \, C_A(Y,t)$. $\square$

<u>Exercise</u>

Generalise L-R bounds to quasi-local Hamiltonians (exponentially-decaying interaction strength).

## Corollary

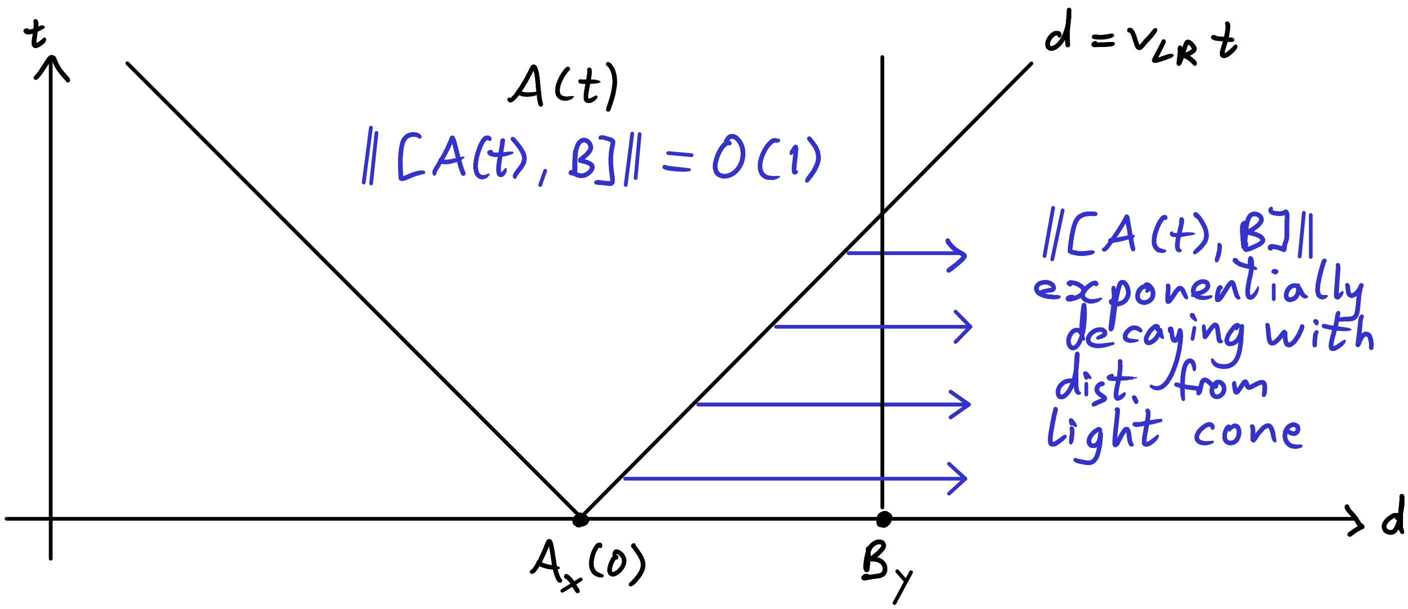
For $k$-local Hamiltonian as in Theorem,

$\exists\, v_{LR}$ depending only on $\mu, s$ s.t.

$$\| [A_X(t), B_Y] \| \leq \min(|X|, |Y|)\, \|A\|\, \|B\|\, e^{-\mu(d(X,Y) - v_{LR} t)}$$

## Proof

Follows immediately from Thm, taking $v_{LR} = \dfrac{2ks}{\mu}$.



$\forall t > 0$, $A_X(t)$ acts non-trivially on entire system.

But L-R bound $\Rightarrow$ can <u>approximate</u> $A(t)$ by observable that only acts within light cone.

For proof, will need

<u>Lemma</u> (Twirling)

$X_{AB} \in B(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$

$$\mathrm{tr}_B[X_{AB}] \otimes \mathbb{1}_B = d_B \int \underbrace{dU}_{\text{Haar measure on } SU(d_B)} \mathbb{1} \otimes U_B \cdot X_{AB} \cdot \mathbb{1} \otimes U_B^\dagger$$

<u>Proof</u>

Note: for any operator $Y$:

$$Y = \sum_{nm} \langle m | Y | n \rangle \cdot |m\rangle\langle n| .$$

Decompose $X_{AB}$ in product basis:

$$X_{AB} = \sum_{ijkl} x_{ijkl} \, \underbrace{|i\rangle\langle j|}_{A} \otimes \underbrace{|k\rangle\langle l|}_{B}$$

so
$$\mathrm{tr}_B X_{AB} = \sum_{ij} \Big( \sum_k x_{ijkk} \Big) |i\rangle\langle j|$$

$$= \sum_{\substack{ijkl \\ mn}} x_{ijkl} \, \delta_{im} \, \delta_{jn} \, \delta_{kl} \; |m\rangle\langle n| \qquad (*)$$

Let $V_B \in SU(d_B)$.

$$\mathbb{1}\otimes V_B \left( d_B \int dU\ \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger \right) \mathbb{1}\otimes V_B^\dagger$$

$$= d_B \int dU\ \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger$$

by invariance of Haar measure

Thus

$$\left[ \mathbb{1}\otimes V_B\ ,\ d_B \int dU\ \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger \right] = 0$$

$$\Rightarrow d_B \int dU\ \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger = Y \otimes \mathbb{1}$$

by Schur's Lemma + fact that unitaries span full algebra $GL(d_B)$.

$$\langle m | Y | n \rangle = \frac{1}{d_B} \operatorname{tr}( |n\rangle\langle m| \otimes \mathbb{1} \cdot Y \otimes \mathbb{1} )$$

$$= \frac{1}{d_B} \operatorname{tr}\Big( |n\rangle\langle m| \otimes \mathbb{1} \cdot d_B \int dU\ \mathbb{1}\otimes U_B \cdot (|i\rangle\langle j| \otimes |k\rangle\langle \ell|) \cdot \mathbb{1}\otimes U_B^\dagger \Big)$$

$$= \delta_{im}\, \delta_{jn} \int dU\ \operatorname{tr}( U |k\rangle\langle \ell| U^\dagger )$$

$$= \delta_{im}\, \delta_{jn}\, \delta_{k\ell}$$

by unitary invariance of trace
+ normalisation of Haar measure $\int dU = 1$

Lemma follows from (*) by linearity. $\square$

**Corollary.**

$\exists\ A_{X(\ell)}(t)$ acting non-trivially only on $X(\ell) = \{ i : d(i, X) \leq v_{LR} t + \ell \}$

$$\| A_X(t) - A_{X(\ell)}(t) \| \leq |X| \, \|A_X\| \, e^{-\mu \ell}$$

**Proof**

Let $A_{X(\ell)}(t) := \int dU \; U A_X(t) U^\dagger$

where $U$ acts on $X(\ell)^c$.

$$A_{X(\ell)}(t) = \mathrm{tr}_{X(\ell)^c} [ A_X(t) ] \otimes \frac{\mathbb{1}}{d_B}$$ by Lemma

$$\| A_X(t) - A_{X(\ell)}(t) \|$$
$$= \int dU \, \| [U, A_X(t)] \|$$ unitary invariance of operator norm
$$\leq \int dU \, |X| \, \|A\| \, e^{-\mu \ell}$$ Lieb-Robinson $+ \|U\| = 1$
$$\leq |X| \, \|A\| \, e^{-\mu \ell} \quad \square$$