

Local Clock Construction

Problem with binary clock is that cannot tell what time it by looking at only a few of the bits \rightarrow non-local H.

Idea: use unary clock.

• Embedding $\iota: \mathbb{C}^{T+1} \hookrightarrow (\mathbb{C}^2)^{\otimes T}$

$$\begin{aligned} \mathbb{C}^{T+1} &\longrightarrow \mathbb{L} \subset (\mathbb{C}^2)^{\otimes T} \\ |t\rangle &\longrightarrow |1\rangle^{\otimes t} |0\rangle^{\otimes T-t} \\ &= \underbrace{|1, \dots, 1\rangle}_t \underbrace{|0 \dots 0\rangle}_{T-t} \end{aligned}$$

(Label unary clock qubits $1, \dots, T$)

• Replace clock terms:

$$|0 \underset{cl}{X} 0\rangle \longrightarrow |0 \underset{1}{X} 0\rangle$$

$$|T \underset{cl}{X} T\rangle \longrightarrow |1 \underset{T}{X} 1\rangle$$

$$|t \underset{cl}{X} t\rangle \longrightarrow |1 \underset{t}{X} 1\rangle \otimes |0 \underset{t+1}{X} 0\rangle \quad t \neq 0, T$$

$$|t \underset{cl}{X} t-1\rangle \longrightarrow |1 \underset{t-1}{X} 1\rangle \otimes |1 \underset{t}{X} 0\rangle \otimes |0 \underset{t+1}{X} 0\rangle \quad t \neq 0, T$$

$$|1 \underset{cl}{X} 0\rangle \longrightarrow |1 \underset{1}{X} 0\rangle \otimes |0 \underset{2}{X} 0\rangle$$

$$|T \underset{cl}{X} T-1\rangle \longrightarrow |1 \underset{T-1}{X} 1\rangle \otimes |1 \underset{T}{X} 0\rangle$$

(Partially) defines a linear mapping on operators

[Super-formally:

Pad mapping to define action on remaining
elems. It X_{sl} & extend by linearity to
 $\mathcal{K} : \mathcal{B}(\mathbb{C}^{T+1}) \mapsto \mathcal{B}((\mathbb{C}^2)^{\otimes T}).$]

Under this mapping,

$$H = H_{in} + H_{prop} + H_{out} \longrightarrow H'$$

where H' acts on unary clock subspace

$\mathcal{L} = \text{span}\{|1\rangle^{\otimes t} |0\rangle^{\otimes T-t}\} \subset (\mathbb{C}^2)^{\otimes T}$ in just the
same way as H acts on original clock \mathbb{C}^{T+1} .

[Super-formally:

$$\begin{array}{ccc} \mathbb{C}^{T+1} & \xrightarrow{H} & \mathbb{C}^{T+1} \\ \downarrow \iota & & \downarrow \iota \\ (\mathbb{C}^2)^{\otimes T} & \xrightarrow{\mathcal{K}(H)} & (\mathbb{C}^2)^{\otimes T} \end{array} \quad]$$

Exercise: Prove this.

What goes wrong if we use
 $|t \times_{cl} t-1\rangle \longrightarrow |1 \times_{t-1} 1\rangle \otimes |1 \times_t 0\rangle$?

Problem: What about "extra" part of Hilbert space? ($\mathcal{L} = (\mathbb{C}^2)^{\otimes T} \setminus \perp(\mathbb{C}^{T+1})$)

Add additional term to Hamiltonian:

$$H_{\text{stab}} = \mathbb{1} \otimes \sum_{t=1}^{T-1} |0\rangle\langle 0|_t \otimes |1\rangle\langle 1|_{t+1}$$

→ gives energy penalty to states not of form $|1, \dots, 1, 0, \dots, 0\rangle$
(i.e. to all states in \mathcal{L}^\perp).

Claim

Same YES/NO-instance bounds for $H' + H_{\text{stab}}$ as for H .

Proof is largely a matter of formalising the blindingly obvious!

Proof

$$\text{Note: } \begin{aligned} \ker H_{\text{stab}} &= \mathcal{L} \supseteq \text{supp } H' \\ \text{supp } H_{\text{stab}} &= \mathcal{L}^\perp \end{aligned}$$

$$H' = \begin{pmatrix} \boxed{H} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{L} \\ \mathcal{L}^\perp \end{matrix}, \quad H_{\text{stab}} = \begin{pmatrix} 0 & 0 \\ 0 & \boxed{P} \end{pmatrix} \begin{matrix} \mathcal{L} \\ \mathcal{L}^\perp \end{matrix}$$

YES instance:

$$\begin{aligned} &\lambda_{\min}(H' + H_{\text{stab}}) \\ &= \min[\lambda_{\min}(H'|_{\mathcal{L}}), \lambda_{\min}(H_{\text{stab}}|_{\mathcal{L}^\perp})] \\ &\quad \text{by above Note} \\ &\leq \lambda_{\min}(H'|_{\mathcal{L}}) = \lambda_{\min}(H) \\ &\leq \frac{\varepsilon}{T+1} \quad \text{as before.} \end{aligned}$$

NO instance:

Note: subspace $\mathcal{L} \subset (\mathbb{C}^2)^{\otimes T}$ invariant under both H' , H_{stab}

(i.e. $\forall |\psi\rangle \in \mathcal{L} : H'|\psi\rangle, H_{\text{stab}}|\psi\rangle \in \mathcal{L}$).

$\Rightarrow H' + H_{\text{stab}}$ decomposes as

$$H' = (H'|_{\mathcal{L}} + H_{\text{stab}}|_{\mathcal{L}}) \oplus (H'|_{\mathcal{L}^\perp} + H_{\text{stab}}|_{\mathcal{L}^\perp})$$

$$\begin{pmatrix} \boxed{H} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \boxed{P} \end{pmatrix} = \begin{pmatrix} \boxed{H} & \\ & \boxed{P} \end{pmatrix}$$

Now

• $(H'|_{\mathcal{L}} + H_{\text{stab}}|_{\mathcal{L}}) = H'|_{\mathcal{L}} + 0 \hat{=} H \geq \Omega\left(\frac{1-\sqrt{\epsilon}}{T^3}\right)$.
from previously

• $(H'|_{\mathcal{L}^\perp} + H_{\text{stab}}|_{\mathcal{L}^\perp}) = 0 + H_{\text{stab}}|_{\mathcal{L}^\perp} = H_{\text{stab}}|_{\text{supp } H_{\text{stab}}}$
 $= \left(\sum_{t=1}^T \mathbb{1} \otimes \pi_t^{(0)} \otimes \pi_{t+1}^{(1)} \right) \Big|_{\text{supp}} \geq 1$.

$\therefore H' + H_{\text{stab}} \geq \Omega\left(\frac{1-\sqrt{\epsilon}}{T^3}\right)$ as before.

□

Overall Hamiltonian is:

$$H' + H_{\text{stab}} = H'_{\text{in}} + \sum_t H'_t + H'_{\text{out}} + H_{\text{stab}}$$

$$H'_{\text{in}} = (\pi_1^{(1)} + \sum_{j \in A} \pi_j^{(1)}) \otimes |0\rangle\langle 0| \quad 2\text{-local}$$

$$H'_{\text{out}} = \pi_1^{(0)} \otimes |1\rangle\langle 1| \quad 2\text{-local}$$

$$H'_t = \frac{1}{2} \mathbb{1} \otimes \left(\overbrace{|1\rangle\langle 1| \otimes |0\rangle\langle 0|}^{2\text{-local}} + \overbrace{|1\rangle\langle 1| \otimes |0\rangle\langle 0|}^{2\text{-local}} \right) \\ - \frac{1}{2} \underbrace{U_{t+1} \otimes (|1\rangle\langle 1| \otimes |1\rangle\langle 0| \otimes |0\rangle\langle 0|)}_{\leq 5\text{-local}} - \text{h.c.}$$

(recall U_t is 1- or 2-qubit gate)

$$H_{\text{stab}} = \sum_{t=1}^{T-1} |0\rangle\langle 0| \otimes |1\rangle\langle 1| \quad 2\text{-local}$$

\therefore Overall Hamiltonian $H' + H_{\text{stab}}$ is 5-local.

Have proven Local Hamiltonian problem is QMA-hard for k -local Hamiltonians with $k \geq 5$!

Exercise: Prove Local Ham. \in QMA.

QED (Kitaev's thm)