QMA-hardness of Local Hamiltonian

Recall:

<u>Def</u> (QMA + amplification)
Decision (YES/NO) problem, problem size n.

Problem $\in$ QMA if
$\exists$ $T = O(\text{poly}(n))$, $\varepsilon = \Omega\left(\frac{1}{\text{poly}(n)}\right)$,
quantum circuit $U = U_T \cdot U_{T-1} \cdots U_2 \cdot U_1$ s.t.

YES:
$\exists\ |w\rangle$: $\Pr(U|w\rangle$ outputs "1") $\geq 1-\varepsilon$

NO:
$\forall\ |w\rangle$: $\Pr(U|w\rangle$ " ") $\leq \varepsilon$

<u>Def</u> (Local Hamiltonian Problem)

Input: k-local Hamiltonian H on n qu<u>d</u>its with m local terms, where $\lambda_0(H) \leq \alpha$ or $\lambda_0(H) \geq \beta$ with $\beta - \alpha \geq \frac{1}{\text{poly}(n)}$

Output: YES if $\lambda_0(H) \leq \alpha$
NO if $\lambda_0(H) \geq \beta$

<u>Thm</u> (Kitaev)
The Local Hamiltonian problem is QMA-hard

We saw that entanglement defeats naive proof approach.

$\longrightarrow$ Need smarter way of encoding witness verification computation into Hamiltonian.

<u>Idea</u> (Feynman): encode evolution of computation in quantum <u>superposition</u>:

"Computational history state" or "history state":

$$|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\psi_t\rangle\, |t\rangle_{cl}$$

computation register $(\mathbb{C}^2)^{\otimes n}$

clock register $\mathbb{C}^{T+1}$

$|\psi_t\rangle$ = state after t steps (gates)

$|\psi_0\rangle$ = initial input state

$|\psi_T\rangle$ = final output state

Clock register takes states $\in \mathbb{C}^{T+1}$:
$|0\rangle, |1\rangle, |2\rangle, \dots, |t\rangle, \dots |T-1\rangle, |T\rangle$

Easy to write down (non-local) Hamiltonian with $|\Psi\rangle$ as ground state:

$$H = H_{in} + H_{prop}$$

$$= (\mathbb{1} - |\psi_0\rangle\langle\psi_0|) \otimes |0\rangle\langle 0|_{cl}$$

ensures correct initial state

$$+ \sum_{t=0}^{T-1} \Big(|\psi_t\rangle\langle\psi_t| \otimes |t\rangle\langle t| + |\psi_{t+1}\rangle\langle\psi_{t+1}| \otimes |t+1\rangle\langle t+1|$$

forces clock to "tick"

$$- |\psi_{t+1}\rangle\langle\psi_t| \otimes |t+1\rangle\langle t|$$

forces 1 step of computation & increments clock

$$- |\psi_t\rangle\langle\psi_{t+1}| \otimes |t\rangle\langle t+1| \Big)$$

necessary for Hermiticity; forces 1 step of computation <u>backwards in time</u> & <u>decrements</u> clock.

<u>Exercise 4</u>

Show that unique g.s. of H is $|\Psi\rangle$.

Challenge is to:

- construct <u>local</u> H for QMA verifier
- prove YES instance $\Rightarrow \lambda_0(H) \leq \alpha$
- prove NO instance $\Rightarrow \lambda_0(H) \geq \beta$

$\left(\beta - \alpha \geq \frac{1}{\text{poly}(n)} \text{ promise gap}\right)$

Proof (Kitaev Thm.)

Hamiltonian (not fully local yet!)

$$H = H_{in} + H_{prop} + H_{out}$$

Initially, take

$$\mathcal{H} = \underbrace{(\mathbb{C}^2)^{\otimes n}}_{\substack{\text{computation}\\\text{register}}} \otimes \underbrace{\mathbb{C}^{T+1}}_{\substack{\text{clock}\\\text{register}}} = \overbrace{\underbrace{\mathbb{C}^2}_{\substack{\text{output}\\\text{qubit}}} \otimes \underbrace{(\mathbb{C}^2)^{\otimes |W|}}_{\substack{\text{witness}\\\text{register}\\\text{qubits } W}} \otimes \underbrace{(\mathbb{C}^2)^{\otimes |A|}}_{\text{ancillas } A}}^{\text{computation register}} \otimes \mathbb{C}^{T+1}$$

(will make clock local later).

$$H_{in} := \Pi_1^{(1)} \otimes |0\rangle\langle 0|_{cl} + \sum_{j \in A} \Pi_j^{(1)} \otimes |0\rangle\langle 0|_{cl}$$

forces output qubit & ancillas to "initially" be in $|0\rangle$ state

$$H_{prop} := \sum_{t=1}^{T-1} H_t, \quad \text{where}$$

$$H_t := \underbrace{\tfrac{1}{2}\, \mathbb{1} \otimes \left(|t\rangle\langle t| + |t+1\rangle\langle t+1|\right)}_{\text{forces clock to "tick"}} \underbrace{- \tfrac{1}{2} U_{t+1} \otimes |t+1\rangle\langle t|}_{\text{forward computation step}} \underbrace{- \tfrac{1}{2} U_{t+1}^{\dagger} \otimes |t\rangle\langle t+1|}_{\text{backwards step}}$$

$$H_{out} := \Pi_1^{(0)} \otimes |T\rangle\langle T|$$

gives energy "penalty" if output of QMA verifier computation is "0".

Lemma

$H_{prop} \sim \mathbb{1} \otimes E$

(i.e. $\exists$ unitary W s.t. $W H_{prop} W^\dagger = \mathbb{1} \otimes E$)

where $E = \sum_t \frac{1}{2} (|t\rangle - |t+1\rangle)(\langle t| - \langle t+1|)$

$$= \begin{pmatrix} 1/2 & -1/2 & & & & \\ -1/2 & 1 & -1/2 & & & \\ & -1/2 & 1 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & 1 & -1/2 \\ & & & & -1/2 & 1/2 \end{pmatrix}$$

Proof

Define unitary

$$W := \sum_{t=0}^{T} \prod_{i=1}^{t} U_i^\dagger \otimes |t\rangle\langle t| \qquad \left(\text{where } \prod_{i=1}^{0} U_i := \mathbb{1}\right)$$

"undoes" t steps of computation

$$= \mathbb{1} \otimes |0\rangle\langle 0| + \sum_{t=1}^{T} (U_1^\dagger \cdot U_2^\dagger \cdots U_{t-1}^\dagger \cdot U_t^\dagger) \otimes |t\rangle\langle t|$$

$W H_t W^\dagger$

$$= \left(U_1^\dagger \cdots U_t^\dagger \otimes |t\rangle\langle t| + U_1^\dagger \cdots U_t^\dagger \cdot U_{t+1}^\dagger \otimes |t+1\rangle\langle t+1|\right)$$

$$\cdot \frac{1}{2}\Big(\mathbb{1} \otimes (|t\rangle\langle t| + |t+1\rangle\langle t+1|)$$

$$- \frac{1}{2} U_{t+1} \otimes |t+1\rangle\langle t| - \frac{1}{2} U_{t+1}^\dagger \otimes |t\rangle\langle t+1| \Big)$$

$$\cdot \left(U_t \cdots U_1 \otimes |t\rangle\langle t| + U_{t+1} \cdot U_t \cdots U_1 \otimes |t+1\rangle\langle t+1|\right)$$

$$= \frac{1}{2} \Big(\overset{=\mathbb{1}}{\cancel{(U_1^\dagger \cdots U_t^\dagger)(U_t \cdots U_1)}} \otimes |t\rangle \overset{=1}{\cancel{\langle t|t\rangle}}$$

$$+ \overset{=\mathbb{1}}{\cancel{(U_1^\dagger \cdots U_t^\dagger U_{t+1}^\dagger)(U_{t+1} U_t \cdots U_1)}} \otimes |t+1\rangle\langle t+1|$$

$$- \overset{=\mathbb{1}}{\cancel{(U_1^\dagger \cdots U_t^\dagger U_{t+1}^\dagger) U_{t+1} (U_t \cdots U_1)}} \otimes |t+1\rangle\langle t|$$

$$- \overset{=\mathbb{1}}{\cancel{(U_1^\dagger \cdots U_t^\dagger)\, U_{t+1}^\dagger \,(U_{t+1} \cdot U_t \cdots U_1)}} \otimes |t\rangle\langle t+1| \Big)$$

$$= \mathbb{1} \otimes |\phi_t\rangle\langle\phi_t| \qquad \text{where } |\phi_t\rangle = \frac{1}{\sqrt{2}}\left(|t\rangle - |t+1\rangle\right)$$

$$H_{prop} \sim W H_{prop} W^\dagger = \sum_t \mathbb{1} \otimes |\phi_t\rangle\langle\phi_t| = \mathbb{1} \otimes E.$$

$\square$

YES instance:

$\exists$ witness $|w\rangle$ s.t.

$\Pr$ (circuit outputs "0" on input $|w\rangle$)
$\leq \varepsilon$ (Def. QMA)

Let $|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\psi_t\rangle |t\rangle_{cl}$

where $|\psi_t\rangle = U_t \cdot U_{t-1} \cdots U_2 \cdot U_1 |\psi_0\rangle$

$$|\psi_0\rangle = \underset{\text{output qubit}}{|0\rangle} \underset{\text{witness}}{|w\rangle} \underbrace{|0\rangle \cdots |0\rangle}_{\text{ancillas}}$$

Note:

- $|\Psi\rangle = \left(\sum_t U_t \cdots U_1 \otimes |t\rangle\langle t|\right)\left(|\psi_0\rangle \otimes \underbrace{\frac{1}{\sqrt{T+1}} \sum_t |t\rangle}_{|\varphi_0\rangle}\right)$
  $= W^\dagger |\psi_0\rangle \otimes |\varphi_0\rangle.$
- $\Pr$(output "0" on input $|w\rangle$)
  $= \langle \psi_T | \Pi_1^{(0)} | \psi_T \rangle \leq \varepsilon.$

Want to show $\lambda_0(H) \leq \alpha$, α small
$\rightarrow$ show computational history state has low energy.

$$
\begin{aligned}
\lambda_0(H) &= \min_{|\phi\rangle} \langle \phi | H | \phi \rangle \\
&\leq \langle \Psi | H | \Psi \rangle \\
&= \langle \Psi | H_{in} | \Psi \rangle + \langle \Psi | H_{prop} | \Psi \rangle + \langle \Psi | H_{out} | \Psi \rangle \\
&= \sim\sim \,''\, \sim\sim + \sum_t \langle \Psi | H_t | \Psi \rangle + \sim\sim \,''\, \sim\sim
\end{aligned}
$$

$$
\begin{aligned}
&\langle \Psi | H_{in} | \Psi \rangle \\
&= \frac{1}{T+1} \left(\sum_{t'=0}^{T} \langle \psi_{t'} | \langle t' | \right) \left(\left(\pi_1^{(1)} + \sum_{j \in anc} \pi_j^{(1)} \right) \otimes |0\rangle\langle 0|_{cL} \right) \cdot \left(\sum_{t=0}^{T} |\psi_t\rangle |t\rangle \right) \\
&\propto \langle \psi_0 | \langle 0 | \left(\left(\pi_1^{(1)} + \sum_{j \in anc} \pi_j^{(1)} \right) \otimes |0\rangle\langle 0|_{cL} \right) |\psi_0\rangle |0\rangle \\
&= \langle 0 | \langle w | \langle 0 |^{\otimes A} \cdot \pi_1^{(1)} \cdot |0\rangle |w\rangle |0\rangle \\
&= \langle 0 | \cancelto{0}{\pi^{(1)}} |0\rangle = 0.
\end{aligned}
$$

$$
\begin{aligned}
&\langle \Psi | H_{prop} | \Psi \rangle \\
&= \langle \psi_0 | \otimes \langle \varphi_0 | \, W \, H_{prop} \, W^{\dagger} \, |\psi_0\rangle \otimes |\varphi_0\rangle \\
&= \langle \psi_0 | \otimes \langle \varphi_0 | \, \mathbb{1} \otimes E \, |\psi_0\rangle \otimes |\varphi_0\rangle \qquad (W H_{prop} W^{\dagger} = \mathbb{1} \otimes E) \\
&= \langle \varphi_0 | \, E \, | \varphi_0 \rangle \\
&\propto \langle \varphi_0 | \sum_t \left(|t\rangle - |t+1\rangle \right) \left(\langle t | - \langle t+1 | \right) \sum_{t'} |t'\rangle \\
&= 0.
\end{aligned}
$$

$$\langle \Psi | H_{out} | \Psi \rangle$$

$$= \frac{1}{T+1} \left(\sum_{t'=0}^{T} \langle \psi_{t'} | \langle t' | \right) \left(\Pi_1^{(0)} \otimes |T\rangle\langle T| \right) \cdot$$

$$\cdot \left(\sum_{t=0}^{T} |\psi_t\rangle |t\rangle \right)$$

$$= \frac{1}{T+1} \langle \psi_T | \Pi_1^{(0)} | \psi_T \rangle$$

$$= \frac{1}{T+1} \Pr(\text{output "0" on input } |\omega\rangle)$$

(from above)

$$\leq \frac{\varepsilon}{T+1}$$

Thus

$$\lambda_0(H) \leq \overset{=0}{\cancel{\langle \Psi | H_{in} | \Psi \rangle}} + \sum_{t=1}^{T} \overset{=0}{\cancel{\langle \Psi | H_t | \Psi \rangle}}$$

$$+ \langle \Psi | H_{out} | \Psi \rangle$$

$$\leq \frac{\varepsilon}{T+1} \quad \text{YES instances}.$$

NO instance:

$\forall\ |w\rangle$:

$\Pr$(circuit outputs "0" on input $|w\rangle$)

$$= \langle \psi_T | \Pi_1^{(0)} | \psi_T \rangle$$

$$\geq 1 - \varepsilon \qquad \text{(Def. QMA)}$$

Need following:

Def (angle between subspaces)

Subspaces $\mathcal{L}_1$, $\mathcal{L}_2$.

$$\cos\theta(\mathcal{L}_1, \mathcal{L}_2) := \max_{\substack{|\varphi_1\rangle \in \mathcal{L}_1 \\ |\varphi_2\rangle \in \mathcal{L}_2}} |\langle \varphi_1 | \varphi_2 \rangle|$$

(cf. $\vec{u} \cdot \vec{v} = \cos\theta$, $|\vec{u}| = |\vec{v}| = 1$)

Lemma (Kitaev)

If $A, B \geq 0$ and
$\lambda_{min}(A|_{\mathrm{supp}A})$, $\lambda_{min}(B|_{\mathrm{supp}B}) \geq \mu$, then

$$A + B \geq 2\mu \sin^2\frac{\theta}{2}$$

where $\theta = \theta(\ker A, \ker B)$.

(Recall $A \geq c \Leftrightarrow A - c\mathbb{1} \geq 0 \Leftrightarrow \lambda_{min}(A) \geq c$.)

Proof

If $\ker A \cap \ker B \neq \emptyset$, $\theta = 0$ & bound is trivially true. So assume wlog $\ker A \cap \ker B = \emptyset$.

Let Π_A = projector onto $\ker A$
Π_B = ——— " ——— $\ker B$

$A \geq \mu (\mathbb{1} - \Pi_A)$, $B \geq \mu (\mathbb{1} - \Pi_A)$ (trivial)

$\rightarrow$ sufficient to prove

$$(\mathbb{1} - \Pi_A) + (\mathbb{1} - \Pi_B) \geq 2 \sin^2 \frac{\theta}{2}$$

or

$$\Pi_A + \Pi_B \leq \overbrace{1 + \cos\theta}^{= 2\cos^2\frac{\theta}{2}} . \qquad (*)$$

Let $|\psi\rangle$ be any eigvect. of $\Pi_A + \Pi_B$ with eigval λ (>0 wlog; $(*)$ trivialy true for $\lambda = 0$):

$$\lambda |\psi\rangle = (\Pi_A + \Pi_B)|\psi\rangle = a|\varphi_A\rangle + b|\varphi_B\rangle$$

where $\Pi_A|\psi\rangle =: a|\varphi_A\rangle$, $\Pi_B|\psi\rangle =: b|\varphi_B\rangle$

& wlog $a, b \in \mathbb{R}^+$ (absorb phases into $|\varphi_{A,B}\rangle$)

$$\lambda = \langle\psi| (\Pi_A + \Pi_B) |\psi\rangle$$

$$= \langle\psi|\Pi_A \Pi_A|\psi\rangle + \langle\psi|\Pi_B \Pi_B|\psi\rangle$$

($\Pi^2 = \Pi$ for projectors)

$$= a^2 + b^2$$

$$\lambda^2 = \langle\psi| (\Pi_A + \Pi_B)^2 |\psi\rangle$$

$$= \underbrace{a^2 + b^2}_{=} + 2ab \, \mathrm{Re}\langle\varphi_A|\varphi_B\rangle$$

$$\leq \lambda + 2ab \, |\mathrm{Re}\langle\varphi_A|\varphi_B\rangle| \qquad a, b \in \mathbb{R}^+$$

$$\leq \lambda + (a^2 + b^2) \, |\mathrm{Re}\langle\varphi_A|\varphi_B\rangle|$$

$$\left(a^2 + b^2 - 2ab = (a-b)^2 \geq 0 \Rightarrow 2ab \leq a^2 + b^2\right)$$

$$= \lambda \left(1 + |\mathrm{Re}\langle\varphi_A|\varphi_B\rangle|\right)$$

$$\leq \lambda \left(1 + \max_{\substack{|\varphi_A\rangle \in \ker A \\ |\varphi_B\rangle \in \ker B}} |\langle\varphi_A|\varphi_B\rangle|\right)$$

$$= \lambda \left(1 + \cos\theta\right)$$

$\therefore \lambda \leq 1 + \cos\theta$ which proves (*). $\square$

Let $A = H_{in} + H_{out}$

$B = H_{prop} = \sum_t H_t$

in Lemma.

Need to bound $\lambda_{min}(A|_{\mathrm{supp}A})$, $\lambda_{min}(B|_{\mathrm{supp}B})$, $\Theta(\ker A, \ker B)$

smallest non-zero eigval

<u>Claim</u> $\lambda_{min}(A|_{\mathrm{supp}A}) \geqslant 1$

<u>Proof</u>: H_{in}, H_{out} commuting projectors

<u>Claim</u> $\lambda_{min}(B|_{\mathrm{supp}B}) \geq \Omega(T^{-2})$

<u>Proof</u>

Define unitary

$$W := \sum_t (U_1^\dagger \cdot U_2^\dagger \cdots U_{t-1}^\dagger \cdot U_t^\dagger) \otimes |t\rangle\langle t|$$

(We used this before in proof of $H \geqslant 0$.)

$$B = H_{prop} \sim W H_{prop} W^\dagger = \mathbb{1} \otimes E$$

where $E = \begin{pmatrix} \frac12 & -\frac12 & & & \\ -\frac12 & 1 & -\frac12 & & \\ & -\frac12 & 1 & \ddots & \\ & & \ddots & \ddots & \\ & & & 1 & -\frac12 \\ & & & -\frac12 & \frac12 \end{pmatrix}$ from before

$\therefore$ Eigvals of B = eigvals of E.

Eigvals & eigvects of E are:

$$\lambda_k = 1 - \cos q_k \quad , \qquad |\varphi_k\rangle \propto \sum_{j=0}^{T} \cos\left(q_k \left(j + \tfrac{1}{2}\right)\right) |j\rangle$$

where

$$q_k = \frac{\pi k}{T+1} \quad , \quad k = 0, \ldots, T$$

<u>Exercise</u> Verify this.

$$\lambda_0 = 0, \quad |\varphi_0\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle .$$

Smallest non-zero eigval

$$\lambda_1 = 1 - \cos\left(\frac{\pi}{T+1}\right) \geq \Omega(T^{-2}). \quad \square$$

Claim $\sin^2\frac{\theta}{2} = \Omega\left(\frac{1-\sqrt{\varepsilon}}{T+1}\right)$, $\theta \equiv \theta(\ker A, \ker B)$

Proof

$$A = H_{in} + H_{out}$$
$$= \left(\Pi_1^{(1)} + \sum_{j\in A} \Pi_j^{(1)}\right) \otimes |0\rangle\langle 0|_{cl} + \Pi_1^{(0)} \otimes |T\rangle\langle T|$$

$$\ker A = \mathrm{span}\{ |0\rangle_1 |\psi\rangle \overbrace{|0\rangle\cdots|0\rangle}^{\text{ancillas}} \otimes |0\rangle_{cl},\ |\varphi\rangle \otimes |t\rangle_{cl}\quad 0<t<T,\ |1\rangle_1 |\phi\rangle \otimes |T\rangle_{cl} \}$$

$$\therefore \Pi_A = |0\rangle\langle 0|_1 \otimes \mathbb{1} \otimes (|0\rangle\cdots|0\rangle)(\langle 0|\cdots\langle 0|) \otimes |0\rangle\langle 0|_{cl} + \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle\langle t| + |1\rangle\langle 1|_1 \otimes \mathbb{1} \otimes |T\rangle\langle T|_{cl}$$

$$W \Pi_A W^\dagger = \overbrace{|0\rangle\langle 0|_1 \otimes \mathbb{1} \otimes (|0\rangle\cdots|0\rangle)(\langle 0|\cdots\langle 0|)}^{\Pi_I} \otimes |0\rangle\langle 0|_{cl} + \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle\langle t|_{cl} + \underbrace{U^\dagger\left(|1\rangle\langle 1|_1 \otimes \mathbb{1}_{[n]\setminus 1}\right) U}_{\Pi_F} \otimes |T\rangle\langle T|_{cl}$$

where $U := U_T U_{T-1} \cdots U_2 U_1$.

$$\cos^2\Theta = \max_{\substack{|\xi\rangle\in\ker A\\ |\eta\rangle\in\ker B}} |\langle\eta|\xi\rangle|^2 \qquad \text{by Def.}$$

$$= \max_{|\xi\rangle,|\eta\rangle} |\langle\eta|\Pi_A|\xi\rangle|^2 \qquad \Pi_A = \text{proj. on } \ker A$$

$$= \max_{|\eta\rangle\in\ker B} \langle\eta|\Pi_A|\eta\rangle \qquad \text{Cauchy-Schwarz}$$

$$\left(\text{saturate } |\langle\eta|\Pi_A|\xi\rangle|^2 \le \|\Pi_A|\eta\rangle\|\cdot\||\xi\rangle\| = \langle\eta|\Pi_A|\eta\rangle\right)$$

$$= \max_{|\eta\rangle\in\ker B} \langle\eta|W^\dagger(W\,\Pi_A\,W^\dagger)\,W|\eta\rangle$$

$$= \max_{|\eta'\rangle\in\ker WBW^\dagger} \langle\eta'|\Big(\Pi_I\otimes|0\rangle\langle 0|_{cl} + \Pi_F\otimes|T\rangle\langle T|_{cl} + \sum_{t=1}^{T-1}\mathbb{1}\otimes|t\rangle\langle t|_{cl}\Big)|\eta'\rangle$$

$$= \frac{1}{T+1}\max_{|\varphi\rangle}\sum_{t=0}^{T}\langle\varphi|\langle t|\big(\sim\big)\sum_{t'=0}^{T}|\varphi\rangle|t'\rangle$$

$$WBW^\dagger = \mathbb{1}\otimes E \Rightarrow |\eta\rangle = |\varphi\rangle\otimes\frac{1}{\sqrt{T+1}}\sum_{t=0}^{T}|t\rangle$$

$$= \frac{1}{T+1}\max_{|\varphi\rangle}\Big(\langle\varphi|(\Pi_I+\Pi_F)|\varphi\rangle + T-1\Big)$$

$$= \frac{1}{T+1}\max_{|\varphi\rangle}\langle\varphi|2\mathbb{1}-\Pi_I^\perp+\Pi_F^\perp|\varphi\rangle + \frac{T-1}{T+1}$$

$$\leqslant \frac{2-2\sin^2\frac{\theta}{2}}{T+1} + \frac{T-1}{T+1} \qquad \text{by Kitaev Lemma}$$

$$= \frac{1+\cos\vartheta}{T+1} + \frac{T-1}{T+1}$$

where $\vartheta = \Theta(\operatorname{supp}\Pi_I, \operatorname{supp}\Pi_F)$.

$$\cos^2\vartheta = \max_{\substack{|\eta\rangle \in \mathrm{supp}\,\Pi_I \\ |\xi\rangle \in \mathrm{supp}\,\Pi_F}} |\langle \eta | \xi \rangle|^2$$

$$= \max_{|w\rangle, |\varphi\rangle} \Big| \underbrace{\left({}_1\langle 0| \langle w| \langle 0| \cdots \langle 0| \right)}_{= |\psi_0\rangle} \cdot \left(U^\dagger |1\rangle_1 |\varphi\rangle \right) \Big|^2$$

$$\Pi_I = |0\rangle_1\langle 0| \otimes \mathbb{1} \otimes (|0\rangle \cdots |0\rangle)(\langle 0| \cdots \langle 0|)$$

$$\Rightarrow |\eta\rangle = |0\rangle_1 |w\rangle |0\rangle \cdots |0\rangle$$

$$\Pi_F = U^\dagger (|1\rangle_1\langle 1| \otimes \mathbb{1}) U$$

$$\Rightarrow |\xi\rangle = |1\rangle_1 |\varphi\rangle$$

$$= \max_{|w\rangle, |\varphi\rangle} (\langle \psi_0 | U^\dagger) \, |1\rangle_1\langle 1| \otimes |\varphi\rangle\langle\varphi| \, (U |\psi_0\rangle)$$

$$\leq \max_{|w\rangle} \langle \psi_T | \Pi_1^{(1)} | \psi_T \rangle$$

$$\leq \varepsilon.$$ NO instance !

$$\therefore \cos^2\theta \leq \frac{1+\cos\vartheta}{T+1} + \frac{T-1}{T+1} \leq 1 - \frac{1-\sqrt{\varepsilon}}{T+1}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\sqrt{\cos^2\theta}}{2} \geq \frac{1-\sqrt{\varepsilon}}{4(T+1)}$$

Taylor expand $\sqrt{1-x} \geq 1-\frac{x}{2}$

Summary:

$$\lambda_{min}\left(A|_{\mathrm{supp}\,A}\right) \geqslant 1$$

$$\lambda_{min}\left(B|_{\mathrm{supp}\,B}\right) \geqslant \Omega(T^{-2})$$

$$\sin^2\frac{\theta}{2} \geqslant \frac{1-\sqrt{\varepsilon}}{4(T+1)}$$

Kitaev's Lemma $\Rightarrow$ for NO instance have

$$H = A + B \geqslant \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right)$$

i.e. have lower bound on λ_{min}:

$$\lambda_{min}(H) = \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right) \quad \text{as required.}$$

We have proven:

$$\text{YES} \Rightarrow \lambda_{min}(H) \leqslant \frac{\varepsilon}{T+1} =: \alpha$$

$$\text{NO} \Rightarrow \lambda_{min}(H) \geqslant \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right) =: \beta$$

Taking $\varepsilon = \Omega\left(\frac{1}{T^2}\right) \Rightarrow \beta - \alpha = \Omega\left(\frac{1}{T^3}\right)$

$\therefore$ Have proven reduction from QMA to Local Hamiltonian ...

... except that clock register has dimension $T+1$

$\longrightarrow$ H not on qudits with const. local dim. d.

Could split clock into $\log T$ qubits
$=$ <u>binary</u> clock $(\mathbb{C}^T \cong (\mathbb{C}^2)^{\otimes \log T})$
$\longrightarrow$ local terms $\log T$-local, not k-local (k const).

(Example of general phenomenon: can always trade off locality against local dimension.)