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1. The spin observable for an arbitrary direction

The operator $J(\theta, \phi)$ corresponding to spin in the spatial direction (θ, ϕ) in spherical coordinates on the unit sphere in three dimensions is

$$J(\theta, \phi) = \sin\theta\cos\phi J_1 + \sin\theta\sin\phi J_2 + \cos\theta J_3.$$

Consider the spin-1/2 representation.
(a) Show that the eigenvalues of $J(\theta, \phi)$ are $\pm\hbar/2$ and that the corresponding normalised eigenvectors may be taken to be

$$|\theta, \phi\rangle = \cos\left(\frac{\theta}{2}\right)\left|\frac{1}{2}\right\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)\left|-\frac{1}{2}\right\rangle,$$

and

$$|\pi - \theta, \phi + \pi\rangle = \sin\left(\frac{\theta}{2}\right)\left|\frac{1}{2}\right\rangle - e^{i\phi}\cos\left(\frac{\theta}{2}\right)\left|-\frac{1}{2}\right\rangle,$$

where $\left|\pm\frac{1}{2}\right\rangle$ satisfy $J_3\left|\pm\frac{1}{2}\right\rangle = \pm\frac{\hbar}{2}\left|\pm\frac{1}{2}\right\rangle$.
(b) Show that the pair of vectors $|\theta, \phi\rangle$ and $|\pi - \theta, \phi + \pi\rangle$ form an orthonormal basis for $\mathbb{C}^2$ (for fixed values of θ, ϕ).
(c) Show also that the operator $J(\theta, \phi)$ may be written in terms of the projectors onto $|\theta, \phi\rangle$ and $|\pi - \theta, \phi + \pi\rangle$ as

$$J(\theta, \phi) = \frac{\hbar}{2}|\theta, \phi\rangle\langle\theta, \phi| - \frac{\hbar}{2}|\pi - \theta, \phi + \pi\rangle\langle\pi - \theta, \phi + \pi|.$$

2. Measurement of spin in arbitrary directions

Consider a spin-1/2 particle in the state $\left|\frac{1}{2}\right\rangle$. By writing the state in terms of the eigenstates of $J(\theta, \phi)$ (defined in question 1), or otherwise, calculate the probability that the eigenvalue $\hbar/2$ is found when $J(\theta, \phi)$ is measured.

3. The uncertainty relations for spin

(a) Derive the following uncertainty relation for spin from the commutation relations:

$$\Delta_{|\psi\rangle}(J_1)\,\Delta_{|\psi\rangle}(J_2) \geq \frac{\hbar}{2}\left|\mathbb{E}_{|\psi\rangle}(J_3)\right|. \tag{1}$$

(note that the right hand side of this equation depends on the state $|\psi\rangle$ unlike the case for the canonical commutation relations). Under what circumstances is there equality in eq. (1)?
(b) Calculate the terms in the uncertainty relation above for the state $|\psi\rangle = \left|\frac{1}{2}\right\rangle$, and confirm that the uncertainty relation is satisfied. Comment on your answer in the light of your answer to the last part of (a).
(c) Calculate $\Delta_{|\psi\rangle}(J_1)$ for the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle + \left|-\frac{1}{2}\right\rangle\right).$$

Is this value consistent with the uncertainty relation?

4. Tensor product operations

Let us define the operators X, Y and Z on $\mathbb{C}^2$ be

$$\begin{aligned} X|0\rangle &= |1\rangle; & X|1\rangle &= |0\rangle \\ Y|0\rangle &= i|1\rangle; & Y|1\rangle &= -i|0\rangle \\ Z|0\rangle &= |0\rangle; & Z|1\rangle &= -|1\rangle. \end{aligned}$$

and let $\mathbb{1}$ denote the identity operator on $\mathbb{C}^2$.

(i) Show that the operator $X \otimes \mathbb{1}$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is unitary. You may use, without proof, the facts that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, and $(A \otimes B)(C \otimes D) = AC \otimes BD$, for operators A, B, C, D.

(ii) Let $|\Psi\rangle$ be the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)$$

on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Calculate $|\Psi_X\rangle = X \otimes \mathbb{1}|\Psi\rangle$.

(iii) Calculate also $|\Psi_Y\rangle = Y \otimes \mathbb{1}|\Psi\rangle$, and $|\Psi_Z\rangle = Z \otimes \mathbb{1}|\Psi\rangle$. Show that the four states $|\Psi\rangle, |\Psi_X\rangle, |\Psi_Y\rangle, |\Psi_Z\rangle$ form an orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$.

5. Correlated measurements on states of two two-level systems

Consider two quantum particles each of which lives in a two dimensional Hilbert space. The particles are in the state

$$|\Psi_0\rangle = |0\rangle|0\rangle,$$

where $|0\rangle$ and $|1\rangle$ are orthonormal basis states for each Hilbert space. The first particle is held by an observer, Alice, and the second by an observer, Bob.

Let X, Y and Z be the Pauli operators.

(i) Write each of the operators X, Y and Z in diagonal form.

(ii) Show that if Alice and Bob both measure Z on their particle (where the pair of particles is in state $|\Psi_0\rangle$), they always get the same answer as each other.

Do their answers always agree if they both measure X or Y?

(iii) Consider now that instead of $|\Psi_0\rangle$ the particles are in the state

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle).$$

Show that if Alice and Bob both measure X, Y or Z, they always get anti-correlated answers.

(iv) More generally calculate the post-measurement state of the particles if Alice measures

$$K(\theta, \phi) = \sin\theta\cos\phi X + \sin\theta\sin\phi Y + \cos\theta Z,$$

on the state $|\Psi_1\rangle$ and finds the eigenvalue 1.

What eigenvalue does Bob find if he now measures the same operator $K(\theta, \phi)$ on his particle [i.e. on the state arising after Alice's measurement]?

Show that whatever eigenvalue Alice finds when she measures $K(\theta, \phi)$, she and Bob always get anti-correlated answers if they both measure the same operator.

6. The Clauser-Horne-Shimony-Holt inequality
As described in the lectures, the CHSH inequality

$$\mathbb{E}(a_1b_1) + \mathbb{E}(a_1b_2) + \mathbb{E}(a_2b_1) - \mathbb{E}(a_2b_2) \leq 2 \tag{2}$$

concerns the experimental situation where a source emits two classical particles, one sent to Alice and one to Bob; Alice and Bob are located sufficiently far apart that no signal can travel from one to the other while the experiment is being performed. Alice has two choices of measurement A_1 and A_2, with outcomes a_1 if she chooses to perform the first measurement and a_2 if she chooses to perform the second. Similarly Bob has two choices of measurements B_1 and B_2, with outcomes denoted b_1 and b_2. We assume that a_1, a_2, b_1, b_2 can only take the values $+1$ or -1.
(i) Consider now that the source emits two quantum particles; one to Alice and one to Bob. The particles are in a state of the form

$$|\Psi_1\rangle = |v\rangle\,|u\rangle\,, \tag{3}$$

where $|v\rangle$ is a state in Alice's Hilbert space and $|u\rangle$ is a state in Bob's. Show there are no choices of local quantum observables A_1, A_2, B_1, B_2 which violate eq. (2).
(ii) Consider now that instead of eq. (3), the source prepares the entangled state

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle\right), \tag{4}$$

where $|0\rangle$ and $|1\rangle$ are orthonormal basis states for the Hilbert space $\mathbb{C}^2$.
Show that for $A_1 = X$, $A_2 = Z$, $B_1 = (X+Z)/\sqrt{2}$, $B_2 = (X-Z)/\sqrt{2}$, the predictions of quantum mechanics violate the CHSH inequality. (The Pauli operators X and Z are defined in question 4).
(iii) Consider now that Alice and Bob both measure one of two operators O_1 or O_2 on their subsystem, where

$$O_1 = \alpha Z + \beta X; \quad O_2 = \gamma Z + \delta X,$$

and $\alpha, \beta, \gamma, \delta$ are complex constants. Find conditions on $\alpha, \beta, \gamma, \delta$ for O_1 and O_2 to be self-adjoint and to have eigenvalues $+1$ and -1.
As in part (ii) of this question, the state of the two particles is $|\Psi_2\rangle$. Calculate the expected values of $O_1 \otimes O_1$, $O_1 \otimes O_2$, $O_2 \otimes O_1$, $O_2 \otimes O_2$, in the state $|\Psi_2\rangle$. Imagine that Alice and Bob must fix $\alpha, \beta, \gamma, \delta$ at the beginning of the experiment and that they can each only measure O_1 or O_2 for these given values of $\alpha, \beta, \gamma, \delta$. Are there any choices of $\alpha, \beta, \gamma, \delta$ (for which O_1 and O_2 are self-adjoint and have eigenvalues $+1$ and -1) for which the expected values violate the CHSH inequality?

[Hint for part (iii): it will help to note that the following inequality holds:

$$|ac + bd| \leq \sqrt{a^2+b^2}\sqrt{c^2+d^2}$$

for any real numbers a, b, c, d.]

7. The Mermin inequality
The Mermin inequality

$$\mathbb{E}(a_1 b_1 c_2) + \mathbb{E}(a_1 b_2 c_1) + \mathbb{E}(a_2 b_1 c_1) - \mathbb{E}(a_2 b_2 c_2) \leq 2 \tag{5}$$

concerns the experimental situation where a source emits three classical particles, one sent to Alice, one to Bob and one to Charlie; Alice, Bob and Charlie are located sufficiently far apart that no signal can travel from one to the other while the experiment is being performed. Alice can measure two properties of her particle; the outcomes are a_1 and a_2. Similarly Bob and Charlie can measure two properties of the particle they receive; the outcomes are b_1 and b_2 (resp. c_1 and c_2). We assume that $a_1, a_2, b_1, b_2, c_1, c_2$ can only be $+1$ or -1.
(i) Prove the Mermin inequality for classical particles. [Hint: consider the case $a_1 = a_2 = 1$, the resulting expression is similar to the CHSH inequality; hence prove the bound for this case. Argue similarly for the other cases.]
(i) Consider now that the source emits three quantum particles; one to each of Alice, Bob and Charlie. The particles are in a state of the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle\,|0\rangle\,|0\rangle + |1\rangle\,|1\rangle\,|1\rangle \right). \tag{6}$$

Consider that Alice, Bob and Charlie measure Y or $-X$. Show that the predictions of quantum mechanics violate the Mermin inequality.