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1. Orbital angular momentum commutation relations

Show that $[AB, C] = A[B, C] + [A, C]B$ for three operators A, B, C. Work out a similar expression for $[AB, CD]$. Hence show that the orbital angular momentum operators L_1, L_2, L_3, where $L_1 = X_2P_3 - X_3P_2$, $L_2 = X_3P_1 - X_1P_3$ and $L_3 = X_1P_2 - X_2P_1$, satisfy the angular momentum commutation relations

$$[L_j, L_k] = i\hbar \sum_{m=1}^{3} \epsilon_{jkm} L_m.$$

(X_j and P_k satisfy the usual canonical commutation relations $[X_j, P_k] = i\hbar\delta_{jk}$.)

2. Raising and lowering operators

Consider three operators J_1, J_2, J_3 satisfying the angular momentum commutation relations. With the definitions $J_\pm = J_1 \pm iJ_2$ and $J^2 = J_1^2 + J_2^2 + J_3^2$, show that

$$\begin{aligned}
[J^2, J_\pm] &= 0; \\
J_+J_- &= J^2 - J_3^2 + \hbar J_3; \\
J_-J_+ &= J^2 - J_3^2 - \hbar J_3; \\
[J_+, J_-] &= 2\hbar J_3; \\
[J_3, J_\pm] &= \pm\hbar J_\pm.
\end{aligned}$$

3. Pauli matrices

The Pauli matrices are defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that

$$\sigma_x^2 = I; \quad \sigma_y^2 = I; \quad \sigma_z\sigma_x = i\sigma_y; \quad \sigma_z\sigma_y = -i\sigma_x;$$

where I is the identity matrix on $\mathbb{C}^2$

(b) Consider two 3-component vectors $\mathbf{A}$ and $\mathbf{B}$ with real entries. Let $\mathbf{A}.\sigma$ and $\mathbf{B}.\sigma$ be the matrices

$$\mathbf{A}.\sigma = A_x\sigma_x + A_y\sigma_y + A_z\sigma_z; \quad \mathbf{B}.\sigma = B_x\sigma_x + B_y\sigma_y + B_z\sigma_z.$$

Write down the matrices $\mathbf{A}.\sigma$ and $\mathbf{B}.\sigma$ explicitly, and hence, or otherwise show that

$$(\mathbf{A}.\sigma)\ (\mathbf{B}.\sigma) = (\mathbf{A}.\mathbf{B})\ I + i(\mathbf{A} \times \mathbf{B}).\sigma.$$

4. The spin one representation

For the $j = 1$ representation of the angular momentum commutation relations, write down the action of the operators $J_\pm$ and J_3 on basis vectors which are simultaneous eigenvectors of J^2 and J_3.
Calculate the action of J_1 and J_2 in this case and hence show that the operators J_1, J_2, J_3 satisfy the angular momentum commutation relations $[J_j, J_k] = i\hbar \sum_{m=1}^{3} \epsilon_{jkm} J_m$.
Calculate the matrices of J_1, J_2, J_3 with respect to the basis you have used above, and show that these matrices also satisfy the angular momentum commutation relations.

5. Matrix elements for angular momentum operators

Let J_1, J_2, J_3 satisfy angular momentum commutation relations, and let $|j,m\rangle$ be the usual (normalised) simultaneous eigenvectors of J^2 and J_3 with eigenvalues $\hbar^2 j(j+1)$ and $m\hbar$ respectively.
By considering $\langle j,m|\, J_+^2 \,|j,m\rangle$ or otherwise, show that

$$\langle j,m|\, J_1^2 \,|j,m\rangle = \langle j,m|\, J_2^2 \,|j,m\rangle \,,$$

and find the value of this matrix element.

6. A spin Hamiltonian

Consider the Hamiltonian

$$H = \frac{1}{2I}(J_1^2 + J_3^2) \ ,$$

where I is a constant.

(i) Show that

$$\langle 1,m|H|1,m\rangle = \frac{1}{4I}(2+m^2)\hbar^2 \ ,$$

for $m = -1, 0, 1$.

(ii) Show also that for $m \neq n$, $\langle 1,m|H|1,n\rangle = 0$ unless $(m,n) = (1,-1)$ or $(m,n) = (-1,1)$. Calculate $\langle 1,1|H|1,-1\rangle$.

(iii) Deduce the three eigenvalues of H and the corresponding eigenstates.