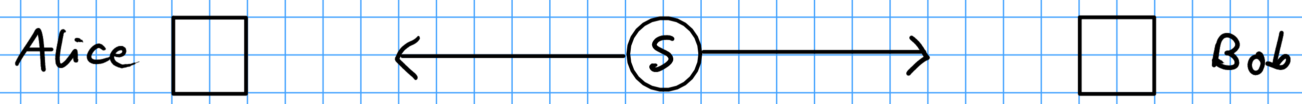
III. Non-Locality and Bell Inequalities

The EPR experiment



- Source sends one particle to Alice, one to Bob.
- Alice & Bob perform measurements on their particles.
- Alice & Bob sufficiently spatially separated such that no signal can travel between them (even at speed of light) in time it takes for Alice & Bob to perform their measurements.

Consider source emitting two particles in state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$

Let Alice measure Z

$\rightarrow$ measurement operator on joint space is $Z \otimes \mathbb{1}$
(Bob isn't doing anything for now)

Outcome $+1 \longrightarrow$ obtains state $|0\rangle|0\rangle$
Outcome $-1 \longrightarrow$ obtains state $|1\rangle|1\rangle$

Now let Bob measure Z

$\rightarrow$ measurement operator on joint space is $\mathbb{1} \otimes Z$

(now Alice isn't doing anything)

$$\mathbb{1} \otimes Z = \underbrace{\mathbb{1} \otimes |0\rangle\langle 0|}_{P_0'} - \underbrace{\mathbb{1} \otimes |1\rangle\langle 1|}_{P_1'}$$

If Alice obtained outcome $+1$, state is $|0\rangle|0\rangle$

$$p(\text{Bob obtains } +1) = \langle 0|\langle 0| \, P_0' \, |0\rangle|0\rangle = 1$$

$$p(\text{" " } -1) = \langle 0|\langle 0| \, P_1' \, |0\rangle|0\rangle = 0$$

If Alice obtained -1, state is now $|1\rangle|1\rangle$

$$p(\text{Bob obtains } +1) = \langle 1|\langle 1| \, P_0' \, |1\rangle|1\rangle = 0$$

$$p(\text{" " } -1) = \langle 1|\langle 1| \, P_1' \, |0\rangle|0\rangle = 1$$

I.e. when Alice & Bob both measure Z, they <u>always</u> obtain identical outcomes.

What if Alice & Bob measure X?

$$X \otimes \mathbb{1} = |+\rangle\langle+| \otimes \mathbb{1} - |-\rangle\langle-| \otimes \mathbb{1}$$
$$\mathbb{1} \otimes X = \mathbb{1} \otimes |+\rangle\langle+| - \mathbb{1} \otimes |-\rangle\langle-|$$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Note that

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle|+\rangle + |-\rangle|-\rangle\right)$$

$\longrightarrow$ same as before, replacing $|0\rangle \leftrightarrow |+\rangle$
$|1\rangle \leftrightarrow |-\rangle$

I.e. Alice & Bob still get identical measurement outcomes if they measure X instead of Z.

According to Q.M., somehow when Alice measures her particle, collapsing its wavefunction, Bob's particle <u>immediately</u> knows which state to collapse into so that he gets identical outcome.

- Einstein very unhappy with this "spooky action at a distance".
- Violates spirit of special relativity (that no signal can travel faster than the speed of light).
- Maybe there's a much simpler explanation...

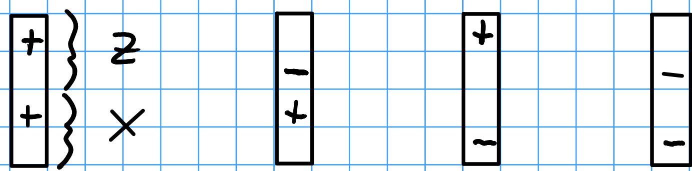
<u>Local hidden variable model</u>

What if in addition to wave function, particles had "hidden" list of "answers" to measurements, and all measurement does is reveal this answer.

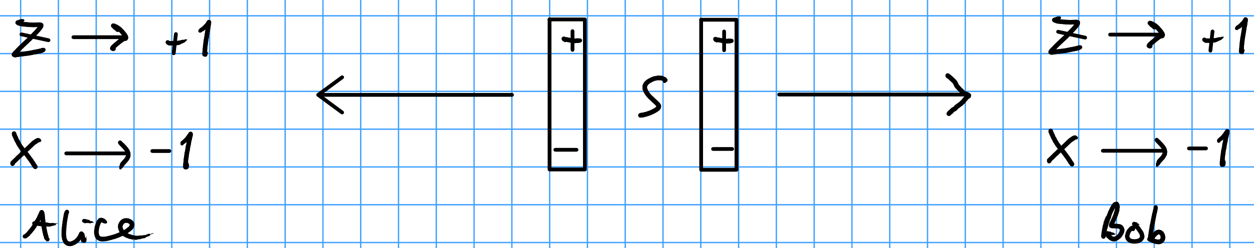
$\boxed{+} \longrightarrow$ measure $+1$ for Z

$\boxed{-} \longrightarrow$ measure -1 for Z

For other measurements, just extend hidden variables to list further measurement outcomes:

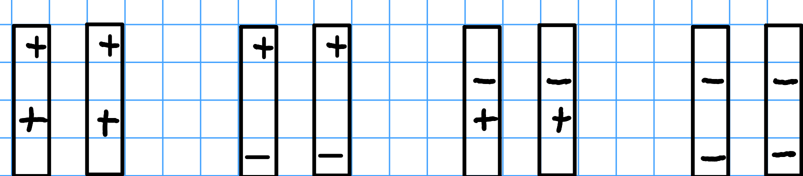


If EPR source produces pairs of particles that carry the same hidden values, e.g.



then explains why they always get same outcome, without any mysterious wavefunction collapse or instantaneous action at a distace (particles carry hidden variables along with them → local).

If source sometimes produces pairs with one set of hidden values, sometimes another set (but always same values for both particles),



also explains why sometimes measure +1, sometimes −1.

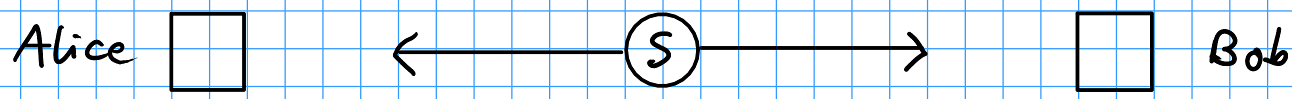
Einstein believed there were hidden variables which, if we knew their values, would allow us to predict the outcomes of measurements exactly, and Q.M. was an effective, statistical theory (much like statistical mechanics describes behiour of gasses probabilistically, though underlying classical mechanics is deterministic).

Bohr believed wave-function collapse and probabilistic measurement outcomes were fundamental part of how Nature works.

No way to resolve who was right, since both explanations were consistent with results of EPR experiment. So there matters stood ...

... until John Bell came up with a beautiful twist on the EPR experiment that allowed hidden variable models to be put to the test!

The Bell Experiment (CHSH version)



- Source sends one particle to Alice, one to Bob.
- Alice has choice of two properties A_1, A_2 that she can measure on her particle, each with two possible values ± 1.
  (Denote the value of property A_1 by $a_1 = \pm 1$, and the value of property A_2 by $a_2 = \pm 1$
- Similarly, Bob has choice of two properties B_1, B_2 with values $b_1, b_2 = \pm 1$.
- Alice & Bob sufficiently spatially separated such that no signal can travel between them during experiment.
- Alice & Bob repeat the experiment many times, choosing at random which property to measure, keeping a record of their measurement choices & outcomes.
- Denote the expected value of e.g. the product $a_1 b_1$ $(= \pm 1)$ by $\mathbb{E}(a_1 b_1)$.
  Note that if they exchange data when experiment is complete, Alice & Bob can estimate $\mathbb{E}(a_1 b_1)$ by taking all runs of experiment in which Alice happened to measure A_1 and Bob B_1, and averaging product $a_1 b_1$ of their outcomes.

Theorem 8 : the CHSH inequality

$$\mathbb{E}(a_1 b_1) + \mathbb{E}(a_1 b_2) + \mathbb{E}(a_2 b_1) - \mathbb{E}(a_2 b_2) \leq 2$$

Proof (simple version)

Consider the quantity

$$C = a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2$$
$$= a_1 (b_1 + b_2) + a_2 (b_1 - b_2)$$

Since $b_1, b_2 = \pm 1$,

either $b_1 + b_2 = \pm 2$ and $b_1 - b_2 = 0$

or $b_1 + b_2 = 0$ and $b_1 - b_2 = \pm 2$

But $a_1, a_2 = \pm 1$ also, so

$$C = a_1 (b_1 + b_2) + a_2 (b_1 - b_2)$$
$$= (\pm 1) \cdot (\pm 2)$$
$$= \pm 2 \leq 2 \qquad (*)$$

Let $P(a_1, a_2, b_1, b_2)$ be the probability that the particles are in the state where property A_1 has the value a_1, A_2 the value a_2, etc.

Then

$$\mathbb{E}(C) = \sum_{a_1 a_2 b_1 b_2} \Big(a_1 (b_1 + b_2) + a_2 (b_1 - b_2) \Big) P(a_1, a_2, b_1, b_2)$$

$$\leqslant \sum_{a_1 a_2 b_1 b_2} 2 \cdot P(a_1, a_2, b_1, b_2) \qquad \text{from } (*)$$

$$= 2$$

But we also have

$$\mathbb{E}(C) = \mathbb{E}(a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2)$$

$$= \mathbb{E}(a_1 b_1) + \mathbb{E}(a_1 b_2) + \mathbb{E}(a_2 b_1) - \mathbb{E}(a_2 b_2)$$

$$\therefore \mathbb{E}(a_1 b_1) + \mathbb{E}(a_1 b_2) + \mathbb{E}(a_2 b_1) - \mathbb{E}(a_2 b_2) \leqslant 2$$

$\square$

Now let's consider what happens if source emits particles in state

$$|\psi\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}},$$

Alice chooses to measure either $A_1 = X$ or $A_2 = Z$, and Bob chooses between $B_1 = \frac{X+Z}{\sqrt{2}}$ or $B_2 = \frac{X-Z}{\sqrt{2}}$.

We will need :

$$\langle\psi| Z \otimes Z |\psi\rangle$$
$$= \frac{1}{2} \left(\langle 0|\langle 0| + \langle 1|\langle 1| \right) Z \otimes Z \left(|0\rangle|0\rangle + |1\rangle|1\rangle \right)$$
$$= \frac{1}{2} (1 + 1) = 1$$

$$\langle\psi| X \otimes X |\psi\rangle$$
$$= \frac{1}{2} \left(\langle 0|\langle 0| + \langle 1|\langle 1| \right) X \otimes X \left(|0\rangle|0\rangle + |1\rangle|1\rangle \right) = 1$$

Now,

$$\mathbb{E}(a_1 b_1) + \mathbb{E}(a_1 b_2) + \mathbb{E}(a_2 b_1) - \mathbb{E}(a_2 b_2)$$
$$= \langle\psi| X \otimes \frac{X+Z}{\sqrt{2}} |\psi\rangle + \langle\psi| X \otimes \frac{X-Z}{\sqrt{2}} |\psi\rangle$$
$$+ \langle\psi| Z \otimes \frac{X+Z}{\sqrt{2}} |\psi\rangle - \langle\psi| Z \otimes \frac{X-Z}{\sqrt{2}} |\psi\rangle$$
$$= \frac{1}{\sqrt{2}} \Big(2 \underbrace{\langle\psi| X \otimes X |\psi\rangle}_{=1} + 2 \underbrace{\langle\psi| Z \otimes Z |\psi\rangle}_{=1} \Big)$$
$$= \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2.$$

But this contradicts Theorem 8 !

i.e. it <u>violates the CHSH inequality</u>

We must have made assumptions in proof of CHSH inequality (Theorem 8) that are not valid for Q.M.

We made two key assumptions in proving CHSH:

1. <u>Locality assumption</u>
   We assumed that, because there is insufficient time for signal to travel from Alice's apporatus to Bob's, Alice's <u>choice</u> of measurement cannot affect Bob's measurement <u>outcome</u>, and vice versa.

2. <u>"Realism" assumption</u>
   We assume that each measured property has a well-defined value, regardless of whether Alice or Bob choose to measure it.

However, the simple proof obscures precisely where and how these assumptions were used.

Proof of CHSH (sophisticated version)

As before

$$\begin{aligned} C &= a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2 \\ &= a_1 (b_1 + b_2) + a_2 (b_1 - b_2) \\ &\leq 2 \qquad (*) \end{aligned}$$

Let $P(a_1\, b_1 \mid A_1\, B_1)$ be the joint probability of obtaining outcomes a_1 and b_1 given that properties A_1 and B_1 were measured.

The marginal

$$P(a_1 \mid A_1, B_1) = \sum_{b_1} P(a_1\, b_1 \mid A_1\, B_1)$$

is then the probability of obtaining outcome a_1 given that A_1 & B_1 were measured.

By the locality assumption, Alice's outcome cannot depend on Bob's choice of measurement, so

$$P(a_1 \mid A_1\, B_1) = P(a_1 \mid A_1\, B_2) \equiv P(a_1 \mid A_1)$$

We also assume that a physical property, A_1 say, has a meaningful value even if Alice in fact chose to measure A_2 (the "realism" assumption). This allows us to legitimately write such things as

$$P(a_1 \mid A_1) = \sum_{a_2} P(a_1\, a_2 \mid A_1)$$

and also e.g.

$$\sum_{a_2} P(a_1 a_2 \mid A_1) = \sum_{a_2} P(a_1 a_2 \mid A_2) \equiv \sum_{a_2} P(a_1 a_2)$$

since, if a_1, a_2 are real, physical values, they should not depend on what Alice happens to choose to measure.

The same arguments apply to all derived probability distributions too, e.g.

$$P(a_1 \mid b_1 A_1 B_1) = P(a_1 \mid b_1 A_1 B_2) \equiv P(a_1 \mid b_1 A_1)$$

Then

$$\mathbb{E}(a_1 b_1) = \sum_{a_1 b_1} a_1 b_1 \; P(a_1 b_1 \mid A_1 B_1)$$

$$= \sum_{a_1 b_1} a_1 b_1 \; P(a_1 \mid A_1 B_1) \cdot P(b_1 \mid a_1 A_1 B_1)$$

[using Bayes' theorem:
$P(xy) = P(x)\, P(y \mid x)$]

$$= \sum_{a_1 b_1} a_1 b_1 \; P(a_1 \mid A_1) \cdot P(b_1 \mid a_1 B_1)$$

[using locality]

$$= \sum_{a_1 b_1} a_1 b_1 \sum_{a_2} \Big(P(a_1 a_2 \mid A_1) \cdot P(b_1 \mid a_1 a_2 B_1) \Big)$$

[using $P(x) = \sum P(xy)$ and "realism"]

$$= \sum_{a_1 b_1} a_1 b_1 \sum_{a_2} \left(P(a_1 a_2 | A_1) \cdot \sum_{b_2} P(b_1 b_2 | a_1 a_2 B_1) \right)$$

[ditto]

$$= \sum_{a_1 b_1} a_1 b_1 \sum_{a_2} \left(P(a_1 a_2) \cdot \sum_{b_2} P(b_1 b_2 | a_1 a_2) \right)$$

[using "realism"]

$$= \sum_{a_1 b_1} a_1 b_1 \; P(a_1, a_2, b_1, b_2)$$

Thus

$$\mathbb{E}(a_1 b_1) + \mathbb{E}(a_1 b_2) + \mathbb{E}(a_2 b_1) - \mathbb{E}(a_2 b_2)$$

$$= \sum_{a_1 b_1 a_2 b_2} (a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2) \cdot P(a_1, a_2, b_1, b_2)$$

$$= \mathbb{E}(C) \le 2 \quad \text{from } (*) \quad \square$$

Quantum mechanics predicts results that violate the CHSH inequality
→ either

- Q. M. violates locality:
  Alice's choice of measurement can affect Bob's outcome, in violation of special relativity (spooky action at a distance)

and / or

- Q. M. violates "realism":
  physical properties have no meaning until we measure them (can't ask whether the moon is there when no one's looking at it!)

This rules out local hidden variable models, so Einstein was wrong.

Bell published the Bell inequality in 1964 (CHSH is a simpler version of the same idea).

In 1981-2, Alain Aspect's group performed the experiment using photons;
the results violated CHSH, in agreement with Q. M.