Maths for Quantum Information

Overview

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Why in English?

Course outline

Lectures	1-4:	Basics of QM & QI	Toby	English
	5-8:	Tensor norms, operator spaces, and Bell inequalities	Nacho	Spanish
	9-12:	Brunn-Minkowski, Levy's Lemma, and quantum thermodynamics	David	"
	13-16:	C^*-algebras	Antonio	"

Assessment

Attend <u>all</u> lectures $\longrightarrow$ oral presentation of research paper

- will offer selection of papers later in course
- can choose to work in pairs (but <u>not</u> > 2)

Skip lectures $\longrightarrow$ EXAM!

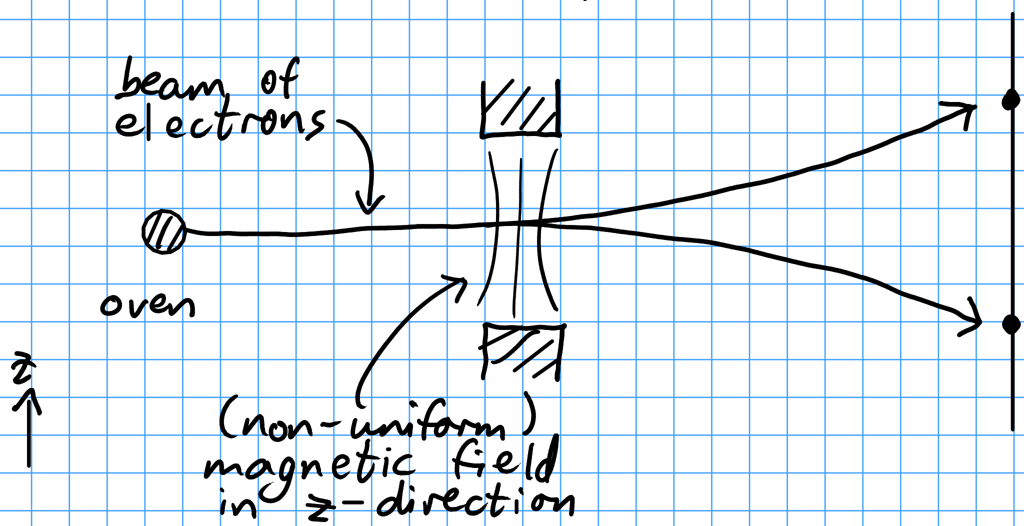
Books for my part of course

"Q. Computation & Q. Information", Nielsen & Chuang (Chapter 2)

"Speakable & Unspeakable in Q.M.", John Bell

The Road to Quantum Weirdness

Stern Gerlach Experiment

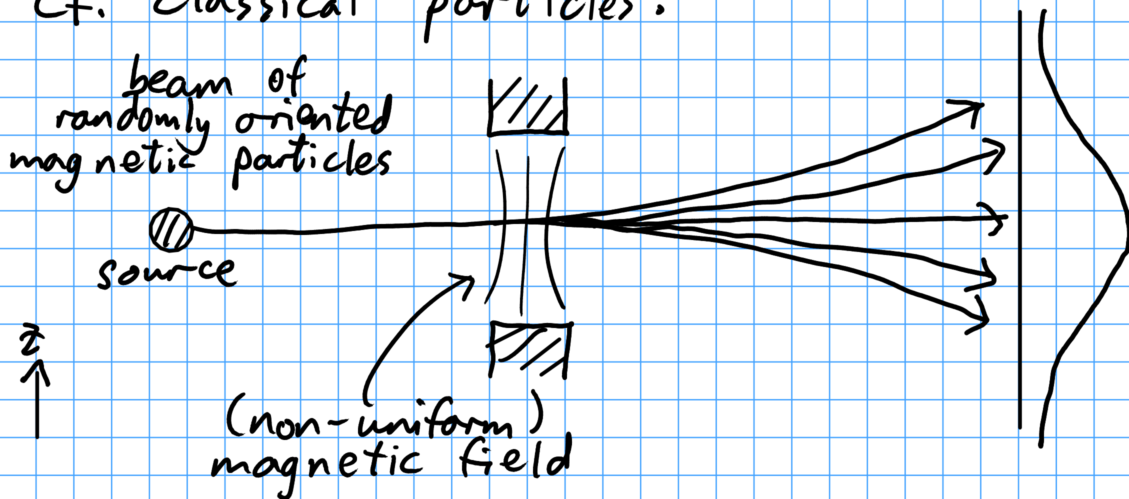


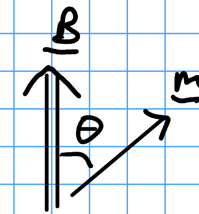
Electrons split into two beams, but <u>never</u> appear in between.

Electrons have intrinsic angular momentum, or "<u>spin</u>", resulting in an intrinsic magnetic moment (it's a charged particle).

S.G. apparatus measures z-component of spin.
But why only two values ?!?

Cf. classical particles:



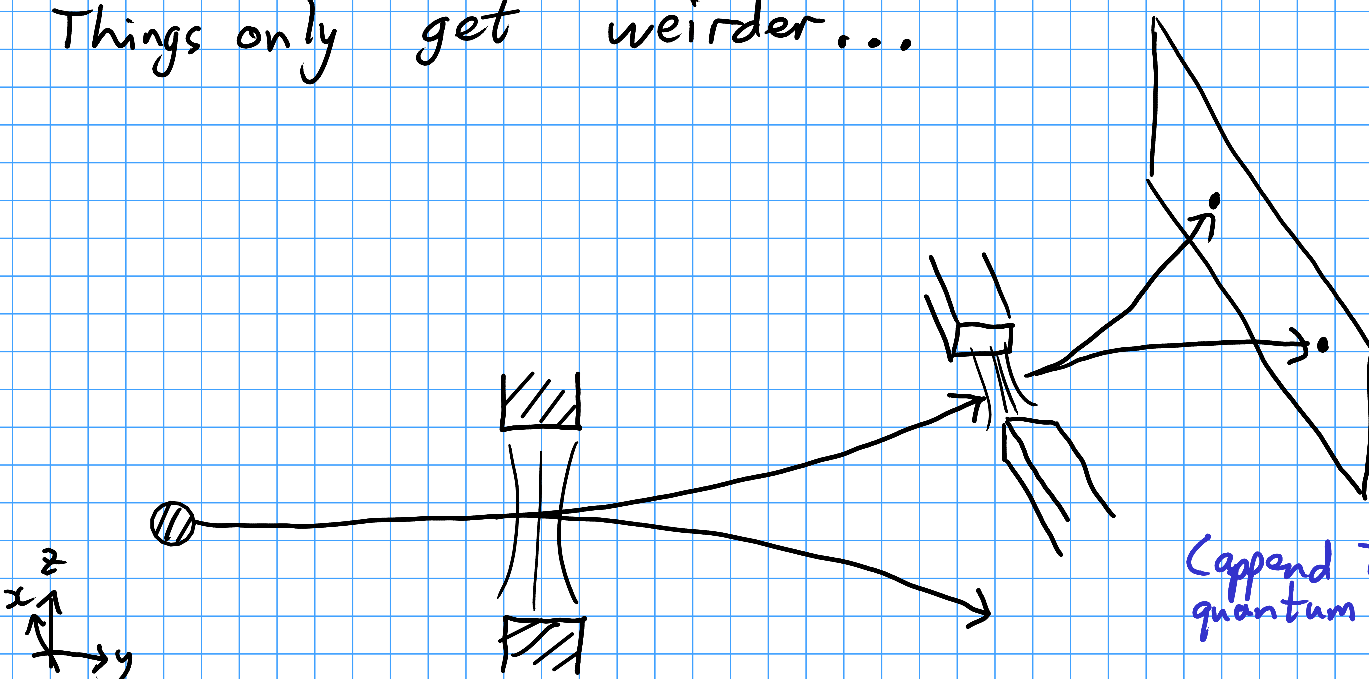


$\underline{m} \cdot \underline{B} = mB\cos\theta$

Deflection depends on component of magnetic moment in z-direction.

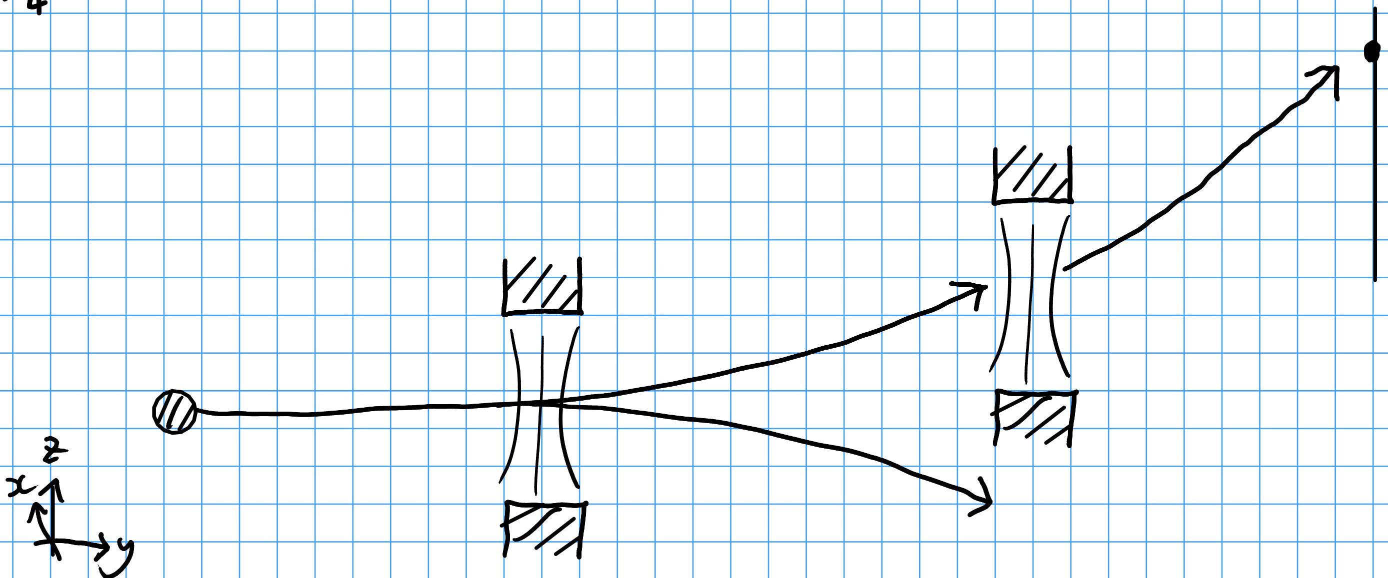
Get distribution of particles due to randomly distributed orientation θ

Things only get weirder...



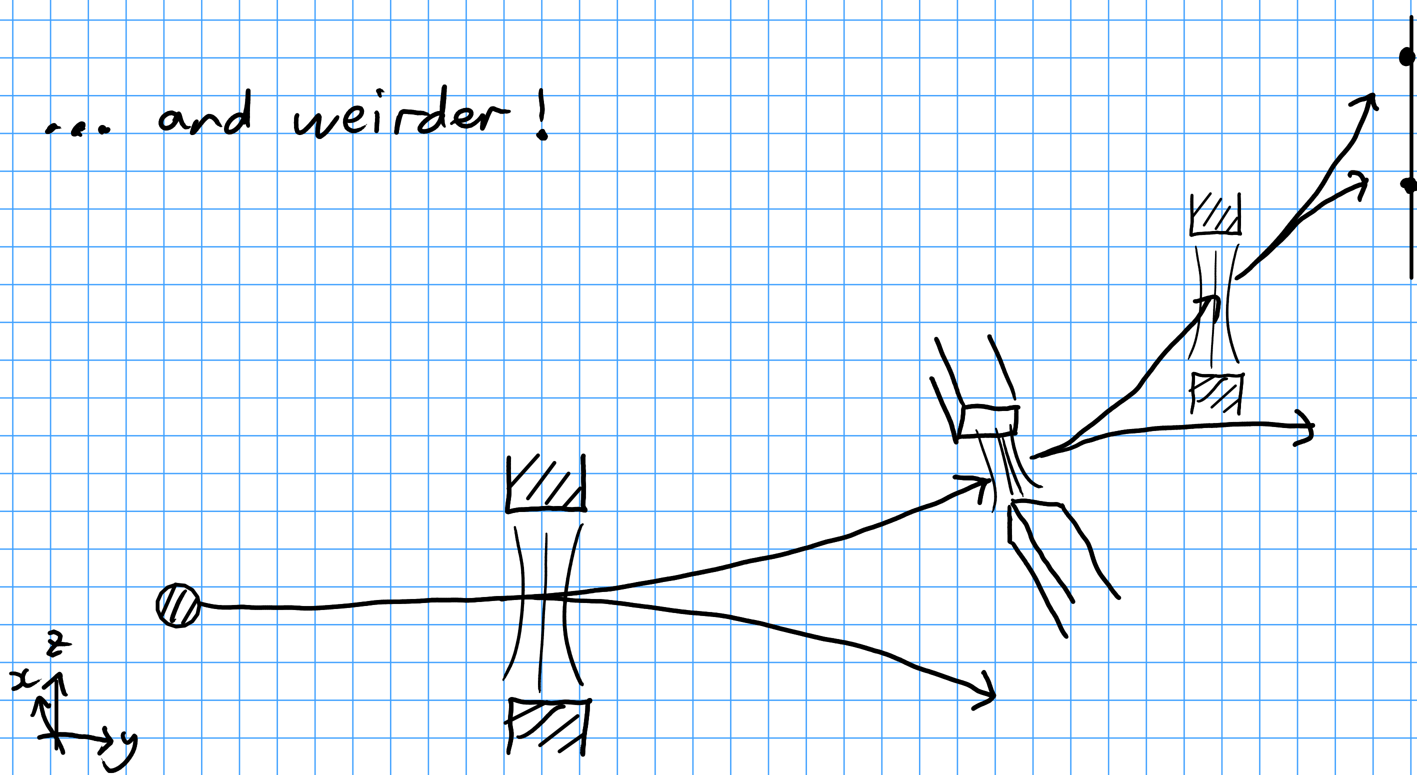
(append to previous quantum diagram)

If take upper beam and feed it through S. G. apparatus oriented along x-axis, beam splits into two (and only two) again!



If we pass one branch through second z-oriented SG, get only one beam as expected.

... and weirder!



But if we put an x-oriented SG in between, second z-oriented SG goes back to splitting beam in two again, as if first one wasn't there!

Clearly, spin doesn't behave like anything we know from classical physics!

Our task for today:
Find a mathematical model that explains S-G.

Note

This is not how Q.M. was discovered historically.
Scientific progress is messy and complicated!
For nice presentation of history of Q.M, see e.g. book "Quantum" by Manjit Kumar.

0. Dirac Notation

Elements (vectors) in Hilbert space $\mathcal{H}$:

$|\psi\rangle \in \mathcal{H}$ called "**kets**".

Elements of dual $\mathcal{H}^*$ (i.e. linear functionals on $\mathcal{H}$):

$\langle\varphi| \in \mathcal{H}^*$ called "**bras**".

Inner product $u(|\varphi\rangle, |\psi\rangle)$:

$\langle\varphi|\psi\rangle \in \mathbb{C}$

Norm

$\| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle}$

Warning!!!

$$u(|\varphi\rangle, \alpha|\psi\rangle + \beta|\phi\rangle) = \langle\varphi|(\alpha|\psi\rangle + \beta|\phi\rangle)$$
$$= \alpha\langle\varphi|\psi\rangle + \beta\langle\varphi|\phi\rangle$$

$$u(\alpha|\varphi\rangle + \beta|\phi\rangle, |\psi\rangle) = (\bar{\alpha}\langle\varphi| + \bar{\beta}\langle\phi|)|\psi\rangle$$
$$= \bar{\alpha}\langle\varphi|\psi\rangle + \bar{\beta}\langle\phi|\psi\rangle$$

so our inner product is <u>linear</u> in <u>2nd argument</u>
<u>conjugate-linear</u> in <u>1st argument</u>

Opposite way around to usual mathematics convention!

Given two states $|\psi\rangle$, $|\varphi\rangle$, we can make an operator

$$|\psi\rangle\langle\varphi|$$

which acts on a state $|\phi\rangle$ as

$$(|\psi\rangle\langle\varphi|)\,|\phi\rangle = \underset{\text{state}}{|\psi\rangle}\;\underset{\text{complex number}}{\langle\varphi|\phi\rangle}$$

In particular, we have projector

$$P_\psi = |\psi\rangle\langle\psi|$$

This gives a convenient notation for self-adjoint operators.

E.g. let $A \in B(\mathbb{C}^2)$ have eigvals λ_1, λ_2 and corresponding eigvects. $|e_1\rangle$, $|e_2\rangle$. Then

$$A = \lambda_1 |e_1\rangle\langle e_1| + \lambda_2 |e_2\rangle\langle e_2|$$

<u>Check</u>:

$$\begin{aligned} A|e_1\rangle &= (\lambda_1 |e_1\rangle\langle e_1|)|e_1\rangle + (\lambda_2 |e_2\rangle\langle e_2|)|e_1\rangle \\ &= \lambda_1 |e_1\rangle (\underbrace{\langle e_1|e_1\rangle}_{=1}) + \lambda_2 |e_2\rangle (\underbrace{\langle e_2|e_1\rangle}_{=0}) \\ &= \lambda_1 |e_1\rangle \end{aligned}$$

as required

eigvects of self-adj. op. are orthogonal

I. The Postulates of Q.M.

1. The state space of a quantum system is a Hilbert space $\mathcal{H}$, and individual states are normalised vectors $|\psi\rangle \in \mathcal{H}$ in that space.

   Everything interesting in Q.I. happens in finite-dimensions! So we will always take $\mathcal{H} = \mathbb{C}^d$.

2. The evolution of a quantum system is described by the Schrödinger equation:

   $$\frac{\partial |\psi\rangle}{\partial t} = -\frac{i}{\hbar} H|\psi\rangle \xrightarrow[\text{solve}]{} |\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = U_t|\psi(0)\rangle$$

   H self-adjoint, called "<u>Hamiltonian</u>" $\longleftarrow$ Physics!

   $\mathcal{H} = \mathbb{C}^d \Rightarrow H$ has matrix representation

   $U_t = e^{-iHt/\hbar}$ <u>unitary operator</u> (given by matrix exponential)

   From now on, $\hbar = c = k = G = 1$ (theorists!)

3. Measurements of a quantum system are described by collection $\{M_m\}$ of measurement operators on its Hilbert space, satisfying completeness:

   $$\sum_m M_m^\dagger M_m = \mathbb{1}$$

   If system is in state $|\psi\rangle$, probability of measuring outcome m is given by:

   $$\Pr(m) = \| M_m |\psi\rangle \| = \langle\psi| M_m^\dagger M_m |\psi\rangle$$

State of the system after obtaining outcome m is:

$$\frac{M_m |\psi\rangle}{\sqrt{\| M_m |\psi\rangle \|}}$$

Note

$$\left\| \frac{M_m |\psi\rangle}{\sqrt{\| M_m |\psi\rangle \|}} \right\| = \frac{\langle \psi | M_m^\dagger M_m | \psi \rangle}{\| M_m |\psi\rangle \|} = 1$$

so post-measurement state is correctly normalised.

Completeness expresses fact that probabilitis sum to 1:

$$1 = \sum_m P_r(m) = \sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle$$

$$= \langle \psi | \sum_m M^\dagger M | \psi \rangle$$

Observation

If $\{ M_m = P_m \}$ are mutually orthogonal projectors, then

$$P_m^\dagger = P_m, \quad P_m^2 = P_m, \quad P_m P_n = 0 \quad m \neq n$$

and Postulate 3 simplifies:

$$P_r(m) = \langle \psi | P_m^\dagger P_m | \psi \rangle = \langle \psi | P_m | \psi \rangle$$

We say $\{ P_m \}$ is a <u>projective measurement</u> (or "von Neumann measurement").

Corollary 1

A projective measurement can be completely described by a single Hermitian operator, which we call an <u>observable</u>.

Proof

Let $A = \sum_m m P_m$. Then $A = A^\dagger$, and the measurement operators $\{P_m\}$ are uniquely defined by the spectral projectors of A.

The eigenvalues m define the measurement outcomes.

Corollary 2

The expectation value of a projective measurement on state $|\psi\rangle$ is given by

$$\langle A \rangle = \langle \psi | A | \psi \rangle \quad \square$$

Proof

Let $A = \sum_m \lambda_m P_m$ be the spectral decomposition of A.

Then

$$\langle A \rangle := \sum_m \Pr(\lambda_m)\, \lambda_m = \sum_m \langle \psi | P_m | \psi \rangle \lambda_m$$

$$= \langle \psi | \sum_m \lambda_m P_m | \psi \rangle = \langle \psi | A | \psi \rangle \quad \square$$

Can anyone see a problem with these postulates?!

Health warning!

Postulate 3 predicts a discontinuous evolution when system is measured (state "collapses" into an eigenstate).

But the measurement apparatus (and ultimately even the laboratory and the experimenter) are made of particles obeying Q.M., i.e. undergoing a continuous evolution (Postulate 2).

Postulates 2 & 3 contradict each other! (The "measurement problem".)

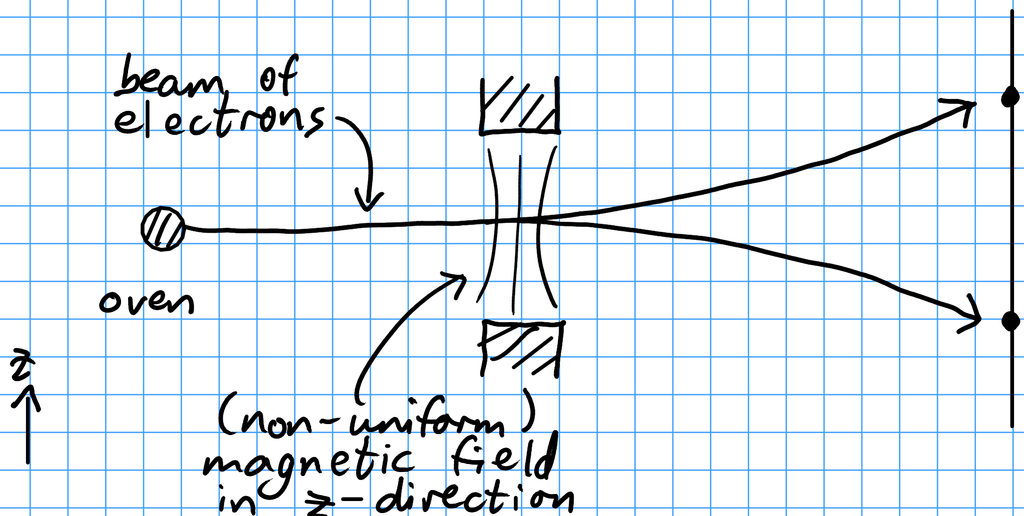
This is the elephant in the closet in Q.M. and we still don't have a good explanation.

What does it all mean?
↳ We still don't have a complete answer.

But we know some things, and they're both fascinating and surprising...

How does this explain the Stern-Gerlach experiment?

<u>Example 1</u>: basic S-G experiment



Why only two beams, nothing in between?

Electrons have "spin" which takes 2 values

- state space spanned by $|0\rangle, |1\rangle$.
- x & z components of spin given by measurement operators

$$z: \{ |0\rangle\langle 0|, |1\rangle\langle 1| \} =: \{ M_0, M_1 \}$$

$$x: \{ |+\rangle\langle +|, |-\rangle\langle -| \} =: \{ M_+, M_- \}$$

where $|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$

$|-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$

Electrons produced with randomly oriented spins
i.e. random state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

α, β random (subject to normalisation $|\alpha|^2+|\beta|^2=1$)

What is the distribution on α, β?
$\longrightarrow$ Haar measure, representation theory
$\longrightarrow$ cf. David's part of the course!

S-G apparatus measures z-component of spin
There are only two possible outcomes,
corresponding to operators $|0\rangle\langle 0|$ & $|1\rangle\langle 1|$.

$$\Pr(\text{outcome } 0) = \| M_0 |\psi\rangle \|$$

$$= \| |0\rangle\langle 0| (\alpha|0\rangle + \beta|1\rangle) \|$$

$$= \| \alpha|0\rangle \| = \langle 0|\bar{\alpha}\alpha|0\rangle = |\alpha|^2$$

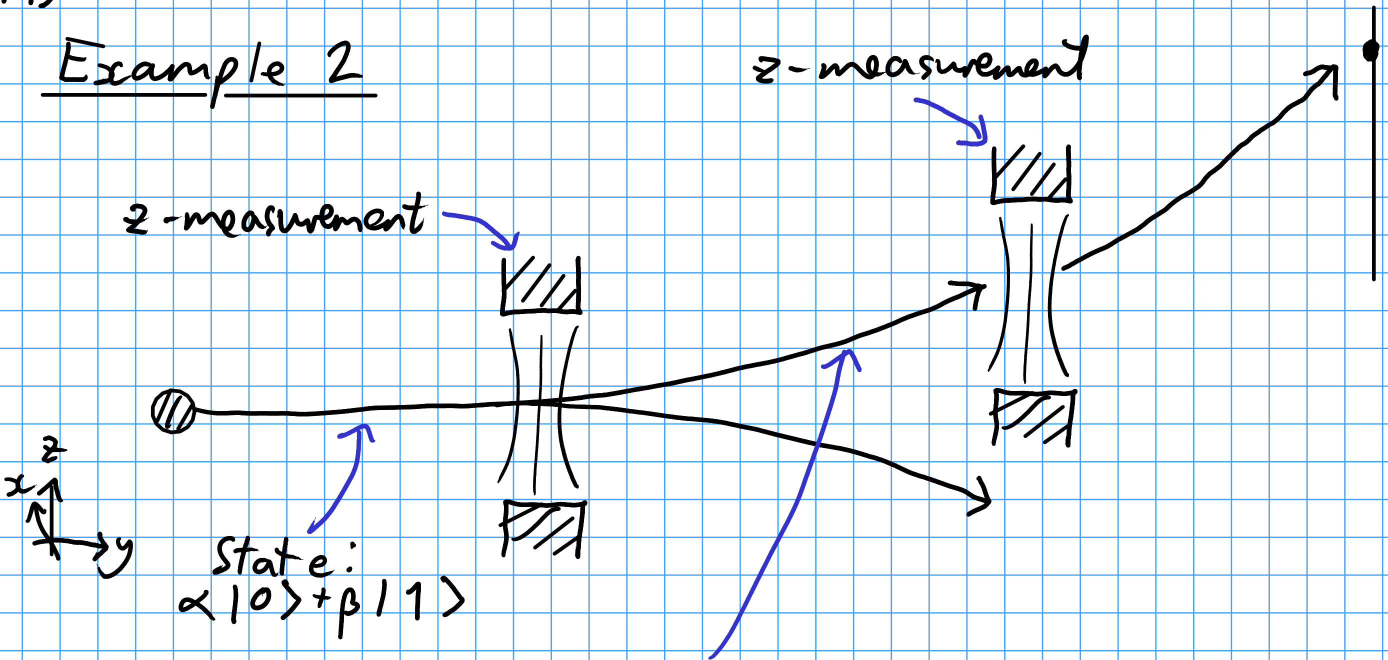
Similarly

$$\Pr(\text{outcome } 1) = |\beta|^2$$

So, S-G apparatus splits beam into two,
one corresponding to outcome 0,
measurement, one to outcome 1.

Averaging over all orientations, i.e. over
random α, β, by symmetry must get equal
numbers of electrons in both beams.

Example 2



Post-measurement state after outcome 0 of (first) z-measurement is

$$\frac{M_0 |\psi\rangle}{\sqrt{\| M_0 |\psi\rangle \|}} = \frac{|0\rangle\langle 0| \cdot (\alpha |0\rangle + \beta |1\rangle)}{|\alpha|} = e^{i\phi} |0\rangle$$

where $\alpha = |\alpha| e^{i\phi}$.

Global phase ϕ has no physical relevance in Q.M., as we're about to see...
(Q. states really live in projective Hilbert space.)

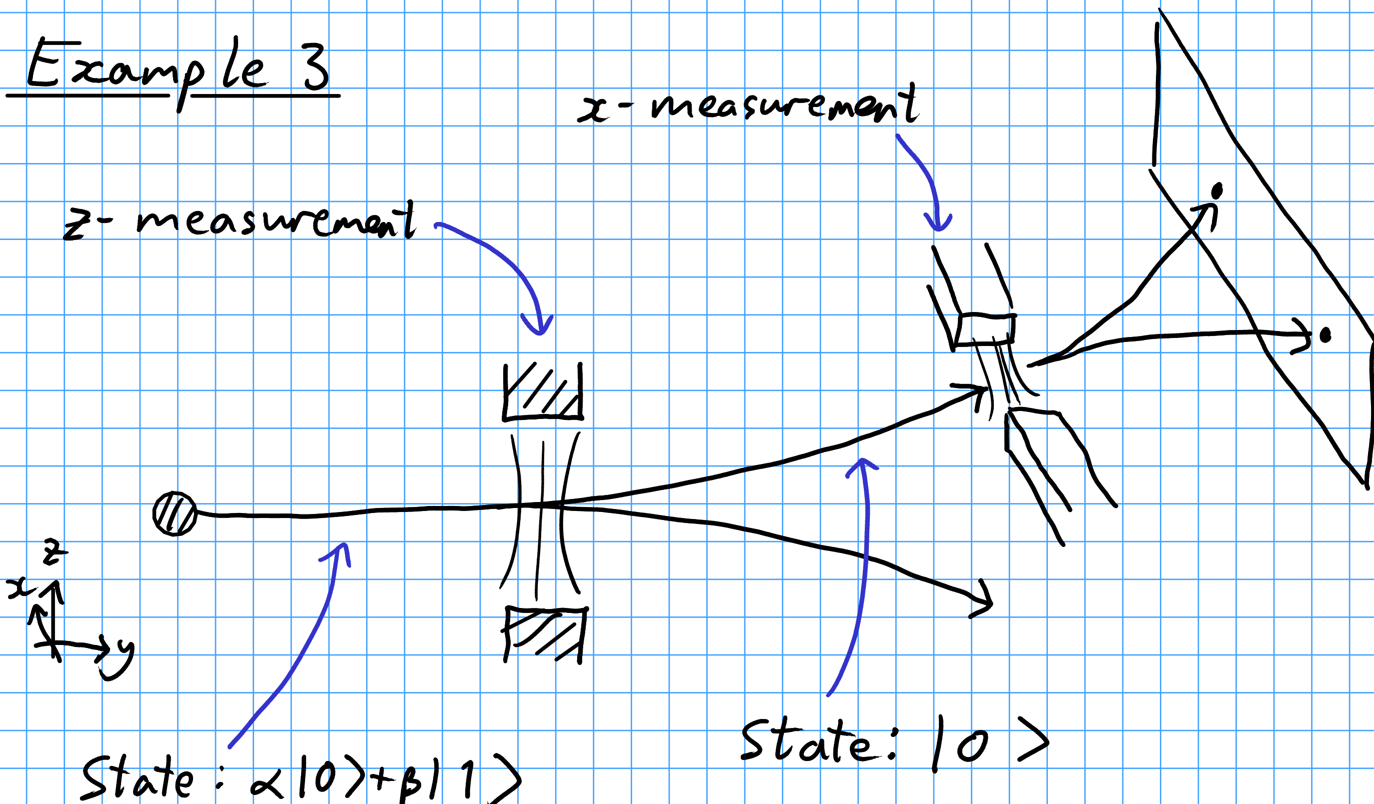
Second z-measurement outcome probabilites:

$$\Pr(0) = \| M_0 e^{i\phi} |0\rangle \| = \langle 0 | e^{-i\phi} |0\rangle\langle 0|^{\dagger} |0\rangle\langle 0| e^{i\phi} |0\rangle$$
$$= \langle 0|0\rangle\langle 0|0\rangle = 1.$$

$$\Pr(1) = \| M_1 e^{i\phi} |0\rangle \| = \langle 1|0\rangle\langle 0|1\rangle = 0.$$

i.e. always get same outcome again (in this case 0)

Example 3



Second (rotated) S-G measures x-component of spin:

Measurement operators $\{ |+\rangle\langle+|, \ |-\rangle\langle-| \}$

$$\Pr(+) = \langle 0 | (|+\rangle\langle+|) |0\rangle = |\langle 0|+\rangle|^2$$
$$= \left| \langle 0 | \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \right|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

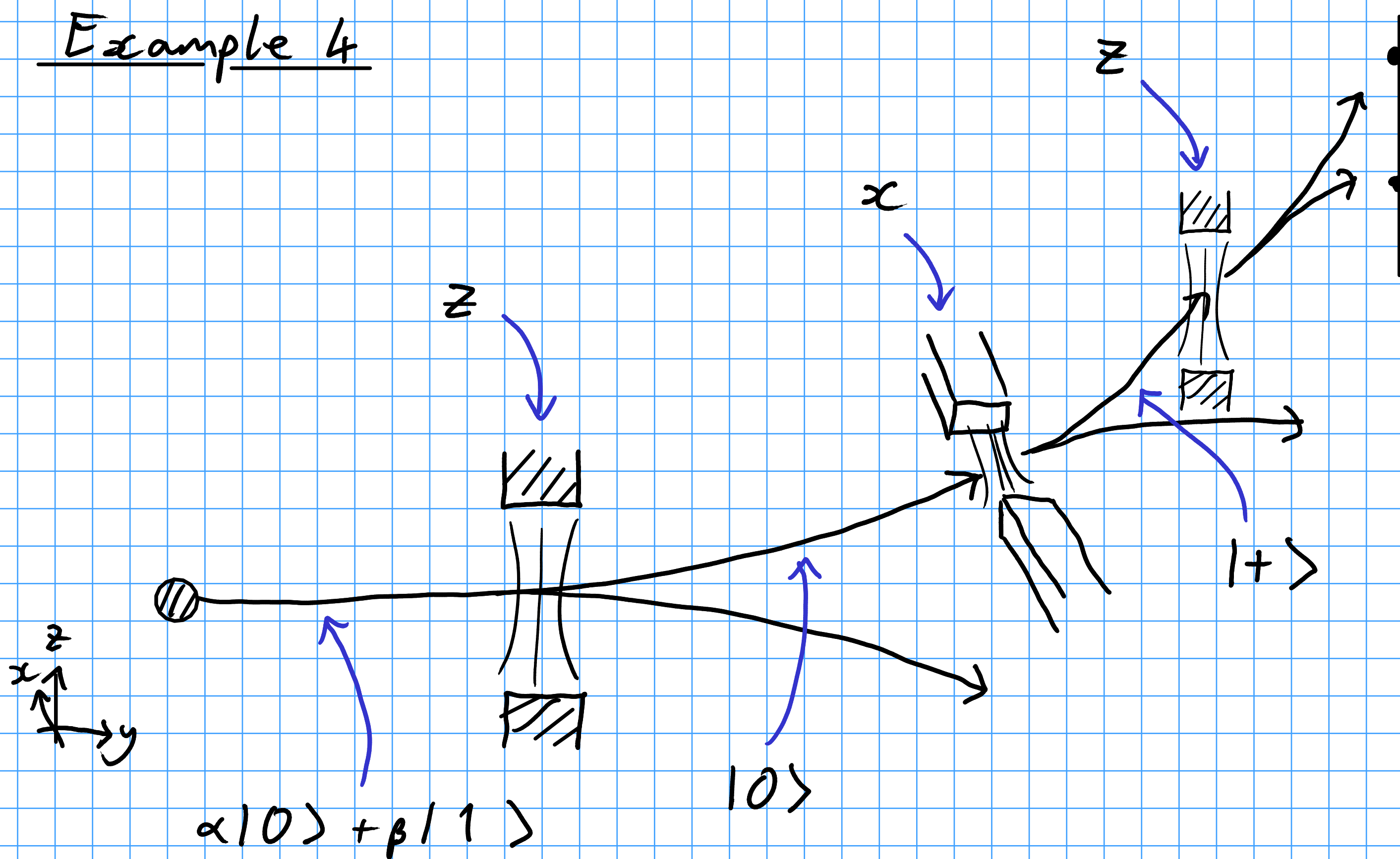
Similarly,

$\Pr(-) = 1/2$.

Thus two outcomes occur with equal probability, so beam <u>splits in two again.</u>

Note that post-measurement state given $+$ outcome is $|+\rangle$, and given $-$ outcome is $|-\rangle$.

Example 4



Final z - measurement of $|+\rangle$ state:

$p(0) = |\langle + | 0 \rangle|^2 = \frac{1}{2}$

$p(1) = |\langle + | 1 \rangle| = \frac{1}{2}$

$\rightarrow$ beam splits again.

Notice how different this is to what we are used to from classical physics (the every day world).

When measure e.g. $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on $|+\rangle$ state, Q.M. says:

- even though know exactly what state is ($|+\rangle$), cannot predict what will happen, just get probabilities of $\frac{1}{2}$, $\frac{1}{2}$ ~~for~~ the two possible outcomes (0 or 1);

- measurement process immediately "collapses" wavefunction into a new state which d ends on what we decide to measure.

↳ Intrinsic unpredictability of Nature

- Is this satisfactory?
  We expect a measurement to reveal a property of the system, and we don't e ect it to completely alter the system

- Einstein was unhappy with this ("I cannot believe that God plays dice").

- Perhaps there are hidden degrees of freedom and if we knew them we could make exact (non-probabilistic) predictions...