I. Hamiltonian Complexity

Notation & Terminology

- $\lambda_0(H)$ or $\lambda_{min}(H) := \min_{|\phi\rangle} \langle\phi| H |\phi\rangle$
  "ground state energy"

- $H|\psi_0\rangle = \lambda_0 |\psi_0\rangle$ $\quad |\psi_0\rangle$ = "ground state"

- $\mathcal{L}_0 = \text{span}\{|\psi\rangle : H|\psi\rangle = \lambda_{min}|\psi\rangle\}$
  $\mathcal{L}_0$ = "g.s. subspace"

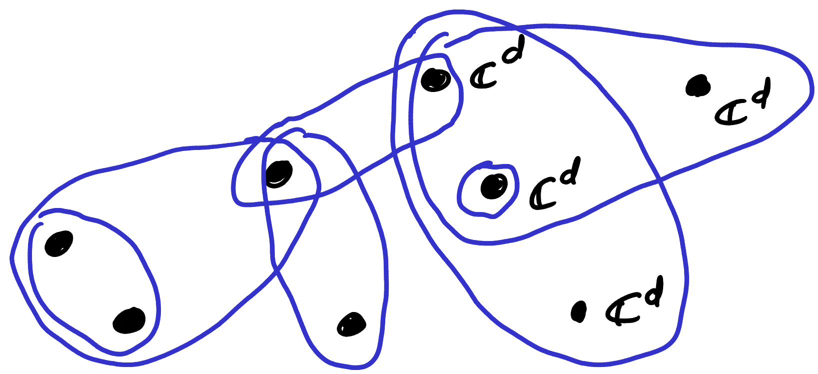
- Note: $H \geq 0$, $\lambda_{min}(H) = 0 \Rightarrow$ g.s. subspace $= \ker H$

- $\Delta(H) =$ "$\lambda_1 - \lambda_0$" $= \min_{|\phi\rangle \perp \mathcal{L}_0} \langle\phi|H|\phi\rangle - \lambda_0(H)$
  "spectal gap": same operator, different eigvals

- $\lambda_0(H_1) - \lambda_0(H_2)$
  "promise gap": different operators, same eigvals

Local Hamiltonians

Core object of study for this <u>entire</u> course.



Many-body quantum system:

- Multipartite Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$
  (often $d=2$: n qubit system)
  e.g. particles with spin-s $\left(d = \frac{s(s+1)}{2}\right)$

- Local interaction Hamiltonian:
  $$h = h_S \otimes \mathbb{1}_{[n]\setminus S}, \quad S \subset [n]$$
  i.e. h acts non-trivially only on subset S of the particles

  We say that interaction h is <u>k-local</u> if $|S| = k$ (i.e. acts non-trivially on k particles).

<u>Notation</u>: often write "h_S" $\equiv h_S \otimes \mathbb{1}_{[n]\setminus S}$, i.e. subscript indicates indices of particles acted on, $\mathbb{1}$ on rest is implicit.

<u>Def</u> (k-local Hamiltonian)

$$H = \sum_i h^{(i)} \in \mathcal{B}((\mathbb{C}^d)^{\otimes n})$$

We say that H is a <u>k-local Hamiltonian</u> (or "H is k-local") if $\forall i \quad h^{(i)}$ is k-local.

I.e. k-local many-body Hamiltonian made of many interactions, each involving at most k particles.

Example: spins on line with nearest-neighbour interactions: $H = \sum_{i=1}^{n-1} h_{i,i+1}$

spins on lattice ——— " ——— ——— " ——— : $H = \sum_{\langle i,j \rangle} h_{ij}$

neighbours

<u>Note:</u> In general, no underlying geometry $\Rightarrow$ no requirement that local interactions are geometrically local.

Physics terminology: "k-local" = "k-body" but stuck with "k-local" now in QIT.

Motivation:

Fundamental interactions in nature (e.g. electromagnetic, or "Coulomb", interaction) are <u>not</u> k-local — interactions decay as $\frac{1}{poly}$, not finite-range.

However, in many-body system electron clouds of neighbouring atoms shield atoms from fields of more distant particles (cf. Faraday cage).

At low-energies, many-body systems often well-approximated by k-local (often 2-local) Hamiltonians.
$\longrightarrow$ k-local H crop up throughout condensed matter physics.

<u>The</u> major topic of cond. mat.: <u>phase transitions</u>.

<u>Quantum</u> phase transitions (e.g. superconductivity, superfluidity) occur at zero or very low temperature characterised by abrupt change in ground state of system

$\longrightarrow$ quantum condensed matter $\approx$ study of ground states of many-body systems.

The "Local Hamiltonian problem" asks whether or not a local Hamiltonian has a low energy ground state.

Def (Local Hamiltonian Problem)

Input: k-local Hamiltonian H on n qudits with m local terms,

"Promise" where $\lambda_0(H) \leq \alpha$ or $\lambda_0(H) \geq \beta$ with $\beta - \alpha \geq \frac{1}{\text{poly}(n)}$

Output: YES if $\lambda_0(H) \leq \alpha$
NO if $\lambda_0(H) \geq \beta$

Note that:

- Input is classical description of H, e.g. specify d^{2k} matrix elements for each of the m terms.
- Wlog $m \leq \binom{n}{k} = O(n^k) = \text{poly}(n)$

  problem size (for fixed k,d) scales as $\text{poly}(n)$

  $\longrightarrow$ take n to measure problem size

- (Strictly speaking, number of bits of data required to specify input also depends on precision of matrix entries. All our results will hold if matrix entries are restricted to poly(n) digits of precision, so we ignore the input precision from now on.)

- Together with input, given promise that condition
  $\lambda_0(H) \leq \alpha$ or $\lambda_0(H) \geq \beta$, $\quad \beta - \alpha = \frac{1}{\mathrm{poly}(n)}$
  holds for the input

  $\longrightarrow$ only required to solve problem under assumption condition holds.

- Conversely, to prove hardness results must <u>show</u> condition <u>does</u> hold for any H we construct.

Thm (Kitaev)
The Local Hamiltonian problem is QMA-hard

The first & most important result in Hamiltonian Complexity theory.

Has many interesting implications for QIT, CompSci, & especially physics (see later)

To prove QMA-hardness:
Transform any QMA prob. $\longrightarrow$ Local H. prob.

Recall Def. QMA:
$\exists$ poly-sized q. circuit $U = U_T \cdot U_{T-1} \cdots U_2 \cdot U_1$ s.t.
YES: $\exists\ |w\rangle : \Pr(U|w\rangle$ outputs "1"$) \geq 2/3$
NO: $\forall\ |w\rangle : \Pr(U|w\rangle$ " " $) \leq 1/3$.

Idea:
"Encode" verifier circuit $\longrightarrow$ local Hamiltonian s.t. ground state encodes output of circuit.

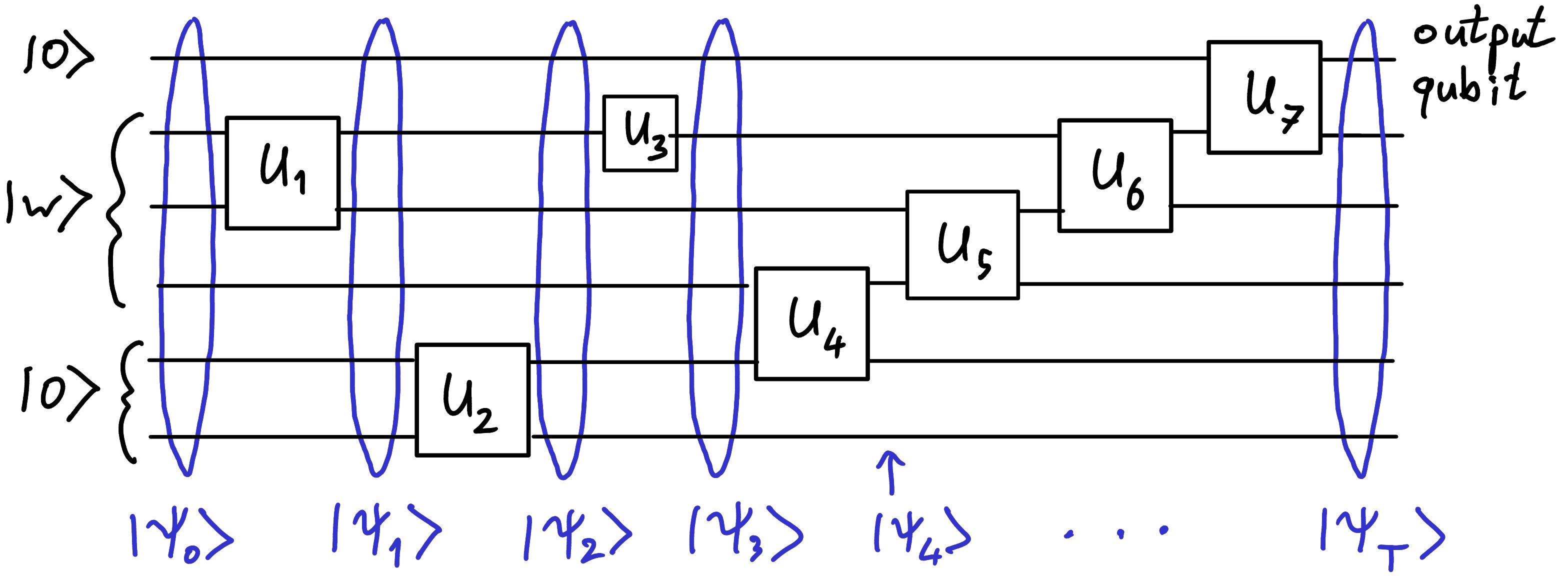
Add term that gives additional energy "penalty" if circuit outputs "0".

YES/NO $\longleftrightarrow$ low-energy/high energy g.s.

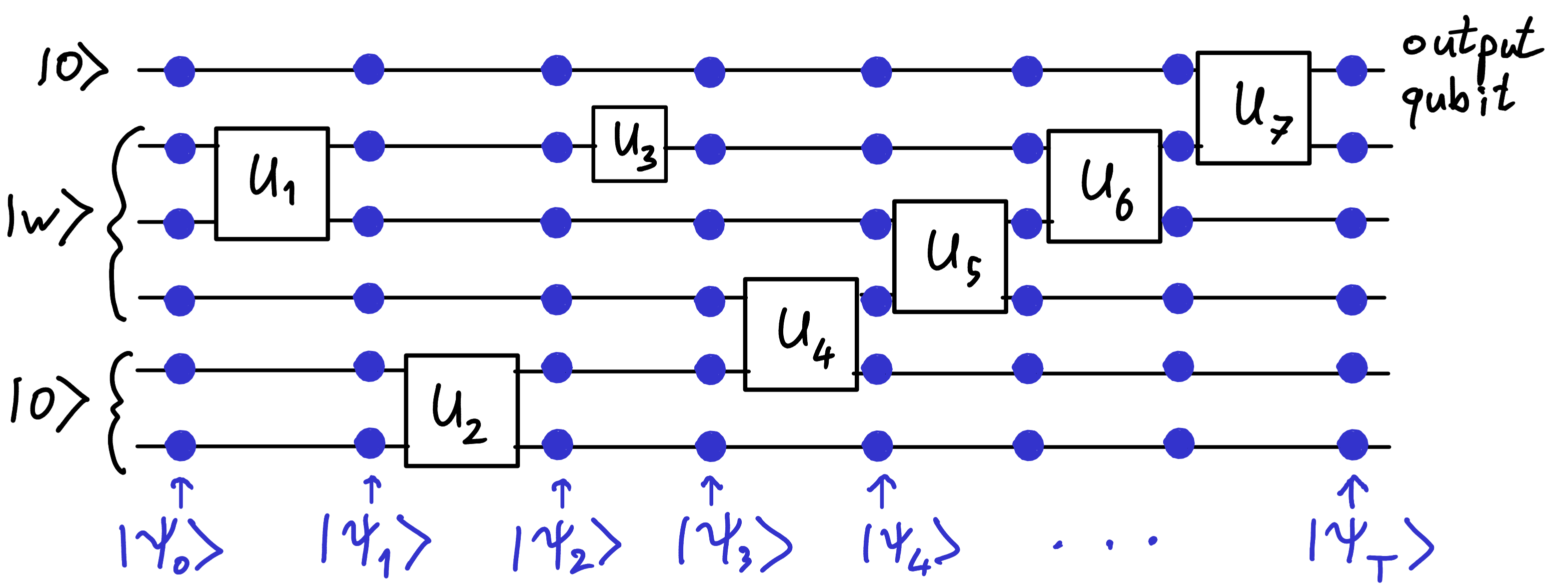
Before proving Thm, we first consider a naive approach that doesn't work, but teaches us something important about quantum many-body states and Hamiltonians.

Naive proof approach (doesn't work!)

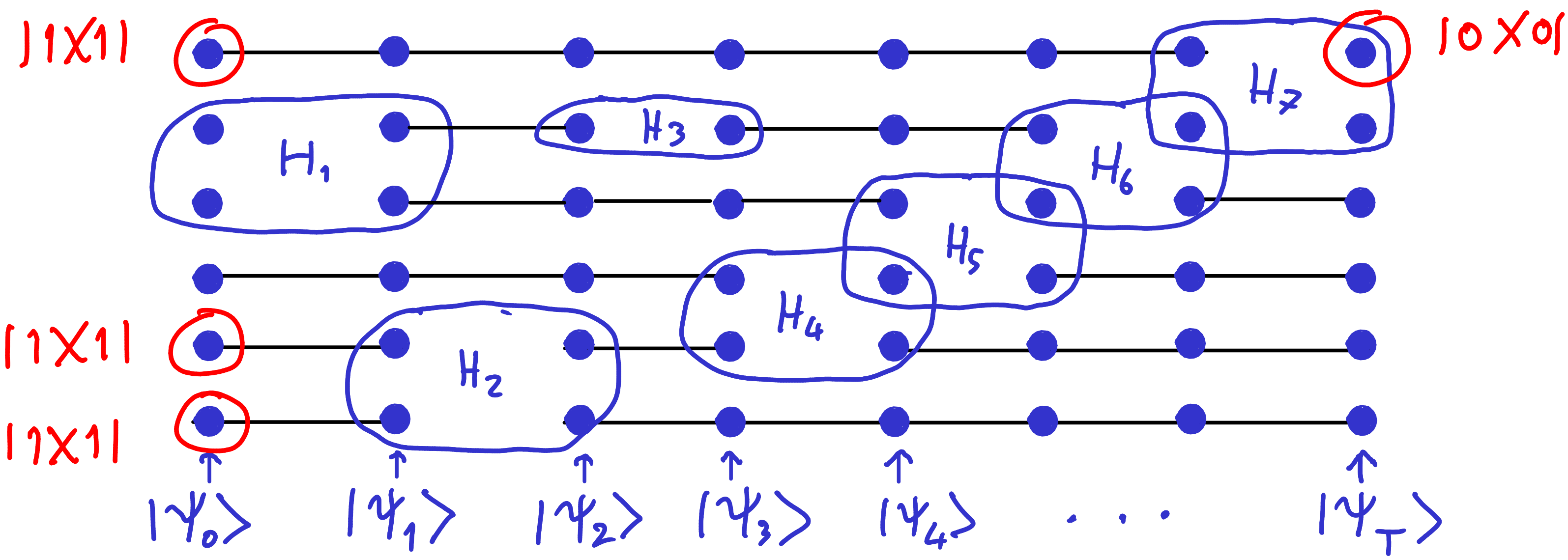
Verifier circuit:



Imagine laying out circuit on lattice of qubits:



Now put local Hamiltonians wherever there's a gate (also need 2-local Hamiltonian terms for identity gates):



Can easily force qubits representing initial state of output & ancilla qubits to be $|0\rangle$ using 1-local term $\pi^{(1)} = |1\rangle\langle 1|$,

But leave initial witness qubits state unconstrained

Similarly, can give additional energy penalty if qubit representing <u>final</u> state of output qubit is "0" using 1-local term $\pi^{(0)}_{out} = |0\rangle\langle 0|$

If we could find Hamiltonian terms $H_1, \ldots, H_T$ s.t. ground states represent correct evolution of circuit, we'd be done:

YES instance:

$\exists$ witness $|w\rangle$ s.t. $\Pr(\text{output } |1\rangle) \geq \frac{2}{3}$

$\Rightarrow \exists$ state $|\psi_0\rangle|\psi_1\rangle \cdots |\psi_T\rangle$ representing correct evolution of circuit s.t. only picks up energy $\leq \frac{1}{3}$ from $\Pi_{out}^{(0)}$

$\longrightarrow$ g.s. energy $\leq \frac{1}{3}$

NO instance:

$\forall \; |w\rangle, \quad \Pr(\text{output } |1\rangle) \leq \frac{1}{3}$

$\Rightarrow \forall \; |\psi\rangle = |\psi_0\rangle|\psi_1\rangle \cdots |\psi_T\rangle$ representing correct evol$^{\underline{n}}$ of circuit, pick up energy $\geq \frac{2}{3}$ from $\Pi_{out}^{(0)}$

$\forall \; |\psi\rangle$ not representing correct evol$^{\underline{n}}$, pick up large energy penalty from $H_1, \ldots, H_T$

$\longrightarrow$ g.s. energy $\geq \frac{2}{3}$

Exercise

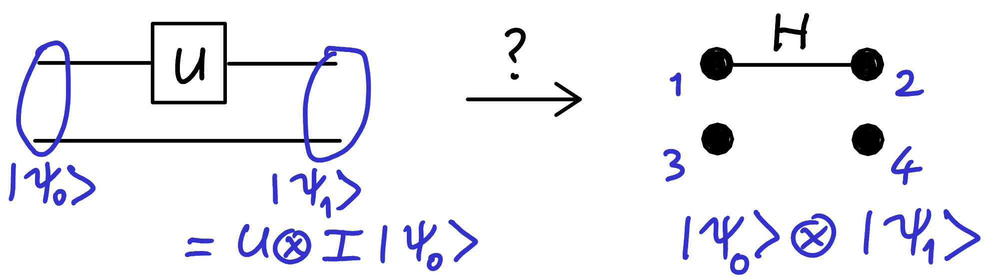
Make above argument rigorous.

<u>If</u> we could find Hamiltonian terms $H_1, \dots, H_T$ s.t. ground states represent correct evolution of circuit, we'd be done.

<u>Exercise</u>

Use this approach to prove <u>NP</u>-hardness of Local Hamiltonian problem for <u>classical</u> Hamiltonians (diagonal in computational basis).

Unfortunately, such Hamiltonian terms cannot exist in quantum case.



<u>Claim</u>

$$\exists U : \nexists H_{12} \text{ s.t. ground state subspace}$$
$$\mathcal{L}_0(H_{12} \otimes \mathbb{1}_{34}) = \text{span}\{ |\psi\rangle_{13} \otimes (U \otimes \mathbb{1}) |\psi\rangle_{24} \}.$$

<u>Proof</u>

Wlog can take $H \geq 0$, $\lambda_{min}(H) = 0$ (substitute $H' = H + \lambda_{min}(H) \cdot I$, eigenvectors unchanged).

$$\longrightarrow \mathcal{L}_0(H \otimes \mathbb{1}) = \ker(H \otimes \mathbb{1}).$$

Consider 2-qubit Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ with trivial gate $U = \mathbb{1}$, so

$$\mathrm{span}\{|\psi\rangle_{13} \otimes (U \otimes \mathbb{1})|\psi\rangle_{24}\} = \mathrm{span}\{|\psi\rangle_{13}|\psi\rangle_{24}\}.$$

Let $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. entangled state! (ebit)

$|\phi\rangle|\phi\rangle \in \ker(H \otimes \mathbb{1})$, so

$$\begin{aligned}
0 &= {}_{13}\langle\phi| \, {}_{24}\langle\phi| \, (H_{12} \otimes \mathbb{1}_{34}) \, |\phi\rangle_{13} |\phi\rangle_{24} \\
&= \mathrm{tr}\left[H_{12} \otimes \mathbb{1}_{34} \cdot |\phi\rangle_{13} |\phi\rangle_{24} \langle\phi|_{13} \langle\phi|_{24} \right] \\
&= \mathrm{tr}\left[H_{12} \cdot \mathrm{tr}_3 (|\phi\rangle\langle\phi|_{13}) \otimes \mathrm{tr}_4 (|\phi\rangle\langle\phi|_{24}) \right] \\
&= \mathrm{tr}\left[H_{12} \cdot \mathbb{1}_1 \otimes \mathbb{1}_2 \right] \\
&= \mathrm{tr}\, H .
\end{aligned}$$

$H \geq 0$, $\mathrm{tr}\, H = 0 \Rightarrow H = 0$.

$$\begin{aligned}
\mathcal{L}_0(H \otimes \mathbb{1}) &= \ker(0 \otimes \mathbb{1}) = \ker(0) \\
&= \mathcal{H} \neq \mathrm{span}\{|\psi\rangle|\psi\rangle\} \qquad \square
\end{aligned}$$

<u>Exercise</u>

Show Proposition holds $\forall\, U$.

QMA-hardness of Local Hamiltonian

Recall:

<u>Def</u> (QMA + amplification)
Decision (YES/NO) problem, problem size n.

Problem $\in$ QMA if
$\exists$ $T = O(\mathrm{poly}(n))$, $\varepsilon = \Omega\left(\frac{1}{\mathrm{poly}(n)}\right)$,
quantum circuit $U = U_T \cdot U_{T-1} \cdots U_2 \cdot U_1$ s.t.

YES:
$\exists$ $|w\rangle$: $\Pr(U|w\rangle$ outputs "1"$) \geq 1-\varepsilon$

NO:
$\forall$ $|w\rangle$: $\Pr(U|w\rangle$ " " $) \leq \varepsilon$

<u>Def</u> (Local Hamiltonian Problem)

Input: k-local Hamiltonian H on n qudits with m local terms, where $\lambda_0(H) \leq \alpha$ or $\lambda_0(H) \geq \beta$ with $\beta - \alpha \geq \frac{1}{\mathrm{poly}(n)}$

Output: YES if $\lambda_0(H) \leq \alpha$
NO if $\lambda_0(H) \geq \beta$

<u>Thm</u> (Kitaev)
The Local Hamiltonian problem is QMA-hard

We saw that entanglement defeats naive proof approach.

$\longrightarrow$ Need smarter way of encoding witness verification computation into Hamiltonian.

<u>Idea</u> (Feynman): encode evolution of computation in quantum <u>superposition</u>:

"Computational history state" or "history state":

$$|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\psi_t\rangle |t\rangle_{cl}$$

computation register $(\mathbb{C}^2)^{\otimes n}$

clock register $\mathbb{C}^{T+1}$

$|\psi_t\rangle$ = state after t steps (gates)

$|\psi_0\rangle$ = initial input state

$|\psi_T\rangle$ = final output state

Clock register takes states $\in \mathbb{C}^{T+1}$:
$|0\rangle, |1\rangle, |2\rangle, \ldots, |t\rangle, \ldots |T-1\rangle, |T\rangle$

Easy to write down (non-local) Hamiltonian with $|\Psi\rangle$ as ground state:

$$H = H_{in} + H_{prop}$$

$$= (\mathbb{1} - |\psi_0\rangle\langle\psi_0|) \otimes |0\rangle\langle 0|_{cl}$$

ensures correct initial state

$$+ \sum_{t=0}^{T-1} \Big(|\psi_t\rangle\langle\psi_t| \otimes |t\rangle\langle t| + |\psi_{t+1}\rangle\langle\psi_{t+1}| \otimes |t+1\rangle\langle t+1|$$

forces clock to "tick"

$$- |\psi_{t+1}\rangle\langle\psi_t| \otimes |t+1\rangle\langle t|$$

forces 1 step of computation & increments clock

$$- |\psi_t\rangle\langle\psi_{t+1}| \otimes |t\rangle\langle t+1| \Big)$$

necessary for Hermiticity; forces 1 step of computation <u>backwards in time</u> & <u>decrements</u> clock.

<u>Exercise 4</u>

Show that unique g.s. of H is $|\Psi\rangle$.

Challenge is to:

- construct <u>local</u> H for QMA verifier
- prove YES instance $\Rightarrow \lambda_0(H) \leq \alpha$
- prove NO instance $\Rightarrow \lambda_0(H) \geq \beta$

$\left(\beta - \alpha \geq \frac{1}{poly(n)}\right.$ promise gap$\left.\right)$

Proof (Kitaev Thm.)

Hamiltonian (not fully local yet!)

$$H = H_{in} + H_{prop} + H_{out}$$

Initially, take

$$\mathcal{H} = \underbrace{(\mathbb{C}^2)^{\otimes n}}_{\text{computation register}} \otimes \underbrace{\mathbb{C}^{T+1}}_{\text{clock register}} = \overbrace{\mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes |W|} \otimes (\mathbb{C}^2)^{\otimes |A|}}^{\text{computation register}} \otimes \mathbb{C}^{T+1}$$

↑ output qubit ↑ witness register qubits W ↑ ancillas A

(will make clock local later).

$$H_{in} := \Pi_1^{(1)} \otimes |0\rangle\langle 0|_{cl} + \sum_{j \in A} \Pi_j^{(1)} \otimes |0\rangle\langle 0|_{cl}$$

forces output qubit & ancillas to "initially" be in $|0\rangle$ state

$$H_{prop} := \sum_{t=1}^{T-1} H_t, \quad \text{where}$$

$$H_t := \frac{1}{2} \mathbb{1} \otimes (|t\rangle\langle t| + |t+1\rangle\langle t+1|)$$

forces clock to "tick"

$$- \frac{1}{2} U_{t+1} \otimes |t+1\rangle\langle t| - \frac{1}{2} U_{t+1}^\dagger \otimes |t\rangle\langle t+1|$$

forward computation step backwards step

$$H_{out} := \Pi_1^{(0)} \otimes |T\rangle\langle T|$$

gives energy "penalty" if output of QMA verifier computation is "0".

Lemma

$H_{prop} \sim \mathbb{1} \otimes E$

(i.e. $\exists$ unitary W s.t. $W H_{prop} W^\dagger = \mathbb{1} \otimes E$)

where $E = \sum_t \frac{1}{2} (|t\rangle - |t+1\rangle)(\langle t| - \langle t+1|)$

$$= \begin{pmatrix} 1/2 & -1/2 & & & & \\ -1/2 & 1 & -1/2 & & & \\ & -1/2 & 1 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & -1/2 \\ & & & & -1/2 & 1/2 \end{pmatrix}$$

Proof

Define unitary

$$W := \sum_{t=0}^{T} \prod_{i=1}^{t} U_i^\dagger \otimes |t\rangle\langle t| \quad \left(\text{where } \prod_{i=1}^{0} U_i := \mathbb{1}\right)$$

"undoes" t steps of computation

$$= \mathbb{1} \otimes |0\rangle\langle 0| + \sum_{t=1}^{T} (U_1^\dagger \cdot U_2^\dagger \cdots U_{t-1}^\dagger \cdot U_t^\dagger) \otimes |t\rangle\langle t|$$

$W H_t W^\dagger$

$$= \left(U_1^\dagger \cdots U_t^\dagger \otimes |t\rangle\langle t| + U_1^\dagger \cdots U_t^\dagger \cdot U_{t+1}^\dagger \otimes |t+1\rangle\langle t+1|\right)$$

$$\cdot \frac{1}{2} \Big(\mathbb{1} \otimes (|t\rangle\langle t| + |t+1\rangle\langle t+1|)$$

$$- \frac{1}{2} U_{t+1} \otimes |t+1\rangle\langle t| - \frac{1}{2} U_{t+1}^\dagger \otimes |t\rangle\langle t+1| \Big)$$

$$\cdot \left(U_t \cdots U_1 \otimes |t\rangle\langle t| + U_{t+1} \cdot U_t \cdots U_1 \otimes |t+1\rangle\langle t+1|\right)$$

$$
\begin{aligned}
= \frac{1}{2} \Big(& \overbrace{\cancel{(U_1^\dagger \cdots U_t^\dagger)(U_t \cdots U_1)}}^{=\mathbb{1}} \otimes |t\rangle \overbrace{\cancel{\langle t|t\rangle}}^{=1} \\
& + \overbrace{\cancel{(U_1^\dagger \cdots U_t^\dagger U_{t+1}^\dagger)(U_{t+1} U_t \cdots U_1)}}^{=\mathbb{1}} \otimes |t+1\rangle\langle t+1| \\
& - \overbrace{\cancel{(U_1^\dagger \cdots U_t^\dagger U_{t+1}^\dagger) U_{t+1} (U_t \cdots U_1)}}^{=\mathbb{1}} \otimes |t+1\rangle\langle t| \\
& - \overbrace{\cancel{(U_1^\dagger \cdots U_t^\dagger) U_{t+1}^\dagger (U_{t+1} U_t \cdots U_1)}}^{=\mathbb{1}} \otimes |t\rangle\langle t+1| \Big)
\end{aligned}
$$

$$
= \mathbb{1} \otimes |\phi_t\rangle\langle\phi_t| \quad \text{where} \quad |\phi_t\rangle = \frac{1}{\sqrt{2}} \left(|t\rangle - |t+1\rangle \right)
$$

$$
H_{prop} \sim W H_{prop} W^\dagger = \sum_t \mathbb{1} \otimes |\phi_t\rangle\langle\phi_t| = \mathbb{1} \otimes E.
$$

$\square$

YES instance:

$\exists$ witness $|w\rangle$ s.t.

Pr(circuit outputs "0" on input $|w\rangle$) $\leq \varepsilon$ (Def. QMA)

Let $|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\psi_t\rangle |t\rangle_{cl}$

where $|\psi_t\rangle = U_t \cdot U_{t-1} \cdots U_2 \cdot U_1 \, |\psi_0\rangle$

$$|\psi_0\rangle = \underset{\text{output qubit}}{|0\rangle} \; \underset{\text{witness}}{|w\rangle} \; \underbrace{|0\rangle \cdots |0\rangle}_{\text{ancillas}}$$

Note:

- $$|\Psi\rangle = \left(\sum_t U_t \cdots U_1 \otimes |t\rangle\langle t|\right)\left(|\psi_0\rangle \otimes \underbrace{\frac{1}{\sqrt{T+1}} \sum_t |t\rangle}_{|\varphi_0\rangle}\right)$$
  $$= W^{\dagger} |\psi_0\rangle \otimes |\varphi_0\rangle.$$
- Pr(output "0" on input $|w\rangle$)
  $$= \langle \psi_T | \Pi_1^{(0)} | \psi_T \rangle \leq \varepsilon.$$

Want to show $\lambda_0(H) \leq \alpha$, α small

$\rightarrow$ show computational history state has low energy.

$$\lambda_0(H) = \min_{|\phi\rangle} \langle \phi | H | \phi \rangle$$

$$\leq \langle \Psi | H | \Psi \rangle$$

$$= \langle \Psi | H_{in} | \Psi \rangle + \langle \Psi | H_{prop} | \Psi \rangle + \langle \Psi | H_{out} | \Psi \rangle$$

$$= \sim\!\sim\, \text{''} \,\sim\!\sim + \sum_t \langle \Psi | H_t | \Psi \rangle + \sim\!\sim\, \text{''} \,\sim\!\sim$$

$$\langle \Psi | H_{in} | \Psi \rangle$$

$$= \frac{1}{T+1} \left(\sum_{t'=0}^{T} \langle \psi_{t'} | \langle t' | \right) \left(\left(\pi_1^{(1)} + \sum_{j \in anc} \pi_j^{(1)} \right) \otimes |0\rangle\langle 0|_{cL} \right) \cdot \left(\sum_{t=0}^{T} |\psi_t\rangle |t\rangle \right)$$

$$\propto \langle \psi_0 | \langle 0 | \left(\left(\pi_1^{(1)} + \sum_{j \in anc} \pi_j^{(1)} \right) \otimes |0\rangle\langle 0|_{cL} \right) |\psi_0\rangle |0\rangle$$

$$= \langle 0 | \langle w | \langle 0 |^{\otimes A} \cdot \pi_1^{(1)} \cdot |0\rangle |w\rangle |0\rangle$$

$$= \langle 0 | \cancel{\pi^{(1)}}^{\,0} |0\rangle = 0.$$

$$\langle \Psi | H_{prop} | \Psi \rangle$$

$$= \langle \psi_0 | \otimes \langle \varphi_0 | \, W \, H_{prop} \, W^{\dagger} \, |\psi_0\rangle \otimes |\varphi_0\rangle$$

$$= \langle \psi_0 | \otimes \langle \varphi_0 | \, \mathbb{1} \otimes E \, |\psi_0\rangle \otimes |\varphi_0\rangle \qquad (W H_{prop} W^{\dagger} = \mathbb{1} \otimes E)$$

$$= \langle \varphi_0 | \, E \, | \varphi_0 \rangle$$

$$\propto \langle \varphi_0 | \sum_t \left(|t\rangle - |t+1\rangle \right) \left(\langle t | - \langle t+1 | \right) \sum_{t'} |t'\rangle$$

$$= 0.$$

$$\langle \Psi | H_{out} | \Psi \rangle$$

$$= \frac{1}{T+1} \left(\sum_{t'=0}^{T} \langle \psi_{t'} | \langle t' | \right) \left(\Pi_1^{(0)} \otimes |T\rangle\langle T| \right) \cdot$$

$$\cdot \left(\sum_{t=0}^{T} |\psi_t\rangle |t\rangle \right)$$

$$= \frac{1}{T+1} \langle \psi_T | \Pi_1^{(0)} | \psi_T \rangle$$

$$= \frac{1}{T+1} \Pr(\text{output "0" on input } |\omega\rangle)$$

(from above)

$$\leq \frac{\varepsilon}{T+1}$$

Thus

$$\lambda_0(H) \leq \cancel{\langle \Psi | H_{in} | \Psi \rangle}^{=0} + \sum_{t=1}^{T} \cancel{\langle \Psi | H_t | \Psi \rangle}^{=0}$$

$$+ \langle \Psi | H_{out} | \Psi \rangle$$

$$\leq \frac{\varepsilon}{T+1} \quad \text{YES instances}.$$

NO instance:

$\forall\ |w\rangle$:

Pr (circuit outputs "0" on input $|w\rangle$)

$= \langle \psi_T | \Pi_1^{(0)} | \psi_T \rangle$

$\geq 1 - \varepsilon$ (Def. QMA)

Need following:

Def (angle between subspaces)

Subspaces $\mathcal{L}_1, \mathcal{L}_2$.

$$\cos\theta(\mathcal{L}_1, \mathcal{L}_2) := \max_{\substack{|\varphi_1\rangle \in \mathcal{L}_1 \\ |\varphi_2\rangle \in \mathcal{L}_2}} |\langle \varphi_1 | \varphi_2 \rangle|$$

(cf. $\vec{u} \cdot \vec{v} = \cos\theta$, $|\vec{u}| = |\vec{v}| = 1$)

Lemma (Kitaev)

If $A, B \geq 0$ and
$\lambda_{min}(A|_{\text{supp}A}), \lambda_{min}(B|_{\text{supp}B}) \geq \mu$, then

$$A + B \geq 2\mu \sin^2\frac{\theta}{2}$$

where $\theta = \theta(\ker A, \ker B)$.

(Recall $A \geq c \Leftrightarrow A - c\mathbb{1} \geq 0 \Leftrightarrow \lambda_{min}(A) \geq c$.)

Proof

If $\ker A \cap \ker B \neq \emptyset$, $\theta = 0$ & bound is trivially true. So assume wlog $\ker A \cap \ker B = \emptyset$.

Let Π_A = projector onto $\ker A$
Π_B = " projector onto " $\ker B$

$A \geq \mu(\mathbb{1} - \Pi_A)$, $B \geq \mu(\mathbb{1} - \Pi_A)$ (trivial)

$\rightarrow$ sufficient to prove

$$(\mathbb{1} - \Pi_A) + (\mathbb{1} - \Pi_B) \geq 2\sin^2\frac{\theta}{2}$$

or

$$\Pi_A + \Pi_B \leq 1 + \cos\theta \,\, = 2\cos^2\frac{\theta}{2}. \qquad (*)$$

Let $|\psi\rangle$ be any eigvect. of $\Pi_A + \Pi_B$ with eigval λ (>0 wlog; (*) trivialy true for $\lambda = 0$):

$$\lambda|\psi\rangle = (\Pi_A + \Pi_B)|\psi\rangle = a|\varphi_A\rangle + b|\varphi_B\rangle$$

where $\Pi_A|\psi\rangle =: a|\varphi_A\rangle$, $\Pi_B|\psi\rangle =: b|\varphi_B\rangle$

& wlog $a, b \in \mathbb{R}^+$ (absorb phases into $|\varphi_{A,B}\rangle$)

$$\lambda = \langle\psi|(\Pi_A + \Pi_B)|\psi\rangle$$
$$= \langle\psi|\Pi_A\Pi_A|\psi\rangle + \langle\psi|\Pi_B\Pi_B|\psi\rangle$$

($\Pi^2 = \Pi$ for projectors)

$$= a^2 + b^2$$

$$
\begin{aligned}
\lambda^2 &= \langle \psi | (\Pi_A + \Pi_B)^2 | \psi \rangle \\
&= \underbrace{a^2 + b^2}_{=\lambda} + 2ab \, \mathrm{Re} \langle \varphi_A | \varphi_B \rangle \\
&\leq \lambda + 2ab \, |\mathrm{Re} \langle \varphi_A | \varphi_B \rangle| \qquad a, b \in \mathbb{R}^+ \\
&\leq \lambda + (a^2 + b^2) \, |\mathrm{Re} \langle \varphi_A | \varphi_B \rangle|
\end{aligned}
$$

$$\left(a^2 + b^2 - 2ab = (a-b)^2 \geq 0 \Rightarrow 2ab \leq a^2 + b^2\right)$$

$$
\begin{aligned}
&= \lambda \left(1 + |\mathrm{Re} \langle \varphi_A | \varphi_B \rangle|\right) \\
&\leq \lambda \left(1 + \max_{\substack{|\varphi_A\rangle \in \ker A \\ |\varphi_B\rangle \in \ker B}} |\langle \varphi_A | \varphi_B \rangle|\right) \\
&= \lambda \left(1 + \cos\theta\right)
\end{aligned}
$$

$\therefore \lambda \leq 1 + \cos\theta$ which proves (*). $\square$

Let $A = H_{in} + H_{out}$

$B = H_{prop} = \sum_t H_t$

in Lemma.

Need to bound $\lambda_{min}(A|_{supp A})$, $\lambda_{min}(B|_{supp B})$, $\Theta(\ker A, \ker B)$

smallest non-zero eigval

<u>Claim</u> $\lambda_{min}(A|_{supp A}) \geq 1$

<u>Proof</u>: H_{in}, H_{out} commuting projectors

<u>Claim</u> $\lambda_{min}(B|_{supp B}) \geq \Omega(T^{-2})$

<u>Proof</u>

Define unitary

$$W := \sum_t (U_1^\dagger \cdot U_2^\dagger \cdots U_{t-1}^\dagger \cdot U_t^\dagger) \otimes |t\rangle\langle t|$$

(We used this before in proof of $H \geq 0$.)

$$B = H_{prop} \sim W H_{prop} W^\dagger = \mathbb{1} \otimes E$$

where
$$E = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & \\ & -\frac{1}{2} & 1 & \ddots & \\ & & \ddots & 1 & -\frac{1}{2} \\ & & & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
from before

$\therefore$ Eigvals of B = eigvals of E.

Eigvals & eigvects of E are:

$$\lambda_k = 1 - \cos q_k \quad , \qquad |\varphi_k\rangle \propto \sum_{j=0}^{T} \cos\left(q_k \left(j + \tfrac{1}{2}\right)\right) |j\rangle$$

where

$$q_k = \frac{\pi k}{T+1} \quad , \quad k = 0, \ldots, T$$

Exercise Verify this.

$$\lambda_0 = 0, \quad |\varphi_0\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle .$$

Smallest non-zero eigval

$$\lambda_1 = 1 - \cos\left(\frac{\pi}{T+1}\right) \geq \Omega(T^{-2}). \quad \square$$

<u>Claim</u> $\sin^2\frac{\theta}{2} = \Omega\left(\frac{1-\sqrt{\varepsilon}}{T+1}\right)$, $\theta \equiv \theta(\ker A, \ker B)$

<u>Proof</u>

$$A = H_{in} + H_{out}$$

$$= \Big(\Pi_1^{(1)} + \sum_{j\in A} \Pi_j^{(1)}\Big) \otimes |0\rangle\langle 0|_{cl} + \Pi_1^{(0)} \otimes |T\rangle\langle T|$$

$$\ker A = \mathrm{span}\Big\{ |0\rangle_1 |\psi\rangle \overbrace{|0\rangle\cdots|0\rangle}^{\text{ancillas}} \otimes |0\rangle_{cl},\ |\varphi\rangle \otimes |t\rangle_{cl} \quad 0<t<T,\ |1\rangle_1 |\phi\rangle \otimes |T\rangle_{cl} \Big\}$$

$$\therefore \Pi_A = |0\rangle\langle 0|_1 \otimes \mathbb{1} \otimes (|0\rangle\cdots|0\rangle)(\langle 0|\cdots\langle 0|) \otimes |0\rangle\langle 0|_{cl} + \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle\langle t| + |1\rangle\langle 1|_1 \otimes \mathbb{1} \otimes |T\rangle\langle T|_{cl}$$

$$W \Pi_A W^\dagger = \overbrace{|0\rangle\langle 0|_1 \otimes \mathbb{1} \otimes (|0\rangle\cdots|0\rangle)(\langle 0|\cdots\langle 0|)}^{\Pi_I} \otimes |0\rangle\langle 0|_{cl} + \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle\langle t|_{cl} + \underbrace{U^\dagger\big(|1\rangle\langle 1|_1 \otimes \mathbb{1}_{[n]\setminus 1}\big)U}_{\Pi_F} \otimes |T\rangle\langle T|_{cl}$$

where $U := U_T U_{T-1} \cdots U_2 U_1$.

$$\cos^2\Theta = \max_{\substack{|\xi\rangle \in \ker A \\ |\eta\rangle \in \ker B}} |\langle \eta | \xi \rangle|^2 \qquad \text{by Def.}$$

$$= \max_{|\xi\rangle, |\eta\rangle} |\langle \eta | \Pi_A | \xi \rangle|^2 \qquad \Pi_A = \text{proj. on } \ker A$$

$$= \max_{|\eta\rangle \in \ker B} \langle \eta | \Pi_A | \eta \rangle \qquad \text{Cauchy-Schwarz}$$

$$\left(\text{saturate } |\langle \eta | \Pi_A | \xi \rangle|^2 \leq \| \Pi_A |\eta\rangle \| \cdot \| |\xi\rangle \| = \langle \eta | \Pi_A | \eta \rangle\right)$$

$$= \max_{|\eta\rangle \in \ker B} \langle \eta | W^\dagger (W \Pi_A W^\dagger) W | \eta \rangle$$

$$= \max_{|\eta'\rangle \in \ker WBW^\dagger} \langle \eta' | \left(\Pi_I \otimes |0\rangle\langle 0|_{cl} + \Pi_F \otimes |T\rangle\langle T|_{cl} + \sum_{t=1}^{T-1} \mathbb{1} \otimes |t\rangle\langle t|_{cl} \right) |\eta'\rangle$$

$$= \frac{1}{T+1} \max_{|\varphi\rangle} \sum_{t=0}^{T} \langle \varphi | \langle t | \left(\sim\sim\sim \right) \sum_{t'=0}^{T} |\varphi\rangle |t'\rangle$$

$$WBW^\dagger = \mathbb{1} \otimes E \Rightarrow |\eta\rangle = |\varphi\rangle \otimes \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle$$

$$= \frac{1}{T+1} \max_{|\varphi\rangle} \left(\langle \varphi | (\Pi_I + \Pi_F) | \varphi \rangle + T - 1 \right)$$

$$= \frac{1}{T+1} \max_{|\varphi\rangle} \langle \varphi | 2\mathbb{1} - \Pi_I^\perp + \Pi_F^\perp | \varphi \rangle + \frac{T-1}{T+1}$$

$$\leq \frac{2 - 2\sin^2\frac{\vartheta}{2}}{T+1} + \frac{T-1}{T+1} \qquad \text{by Kitaev Lemma}$$

$$= \frac{1 + \cos\vartheta}{T+1} + \frac{T-1}{T+1}$$

where $\vartheta = \Theta(\operatorname{supp} \Pi_I, \operatorname{supp} \Pi_F)$.

$$\cos^2\vartheta = \max_{\substack{|\eta\rangle \in \operatorname{supp}\Pi_I \\ |\xi\rangle \in \operatorname{supp}\Pi_F}} |\langle \eta | \xi \rangle|^2$$

$$= \max_{|w\rangle, |\varphi\rangle} \Big| \big(\underbrace{{}_1\langle 0| \langle w| \langle 0| \cdots \langle 0|}_{= |\psi_0\rangle}\big) \cdot \big(U^\dagger |1\rangle_1 |\varphi\rangle\big) \Big|^2$$

$$\Pi_I = |0\rangle_1\langle 0| \otimes \mathbb{1} \otimes (|0\rangle \cdots |0\rangle)(\langle 0| \cdots \langle 0|)$$

$$\Rightarrow |\eta\rangle = |0\rangle_1 |w\rangle |0\rangle \cdots |0\rangle$$

$$\Pi_F = U^\dagger (|1\rangle_1\langle 1| \otimes \mathbb{1}) U$$

$$\Rightarrow |\xi\rangle = |1\rangle_1 |\varphi\rangle$$

$$= \max_{|w\rangle, |\varphi\rangle} (\langle \psi_0 | U^\dagger)\, |1\rangle_1\langle 1| \otimes |\varphi\rangle\langle\varphi|\, (U |\psi_0\rangle)$$

$$\leq \max_{|w\rangle} \langle \psi_T | \Pi_1^{(1)} | \psi_T \rangle$$

$$\leq \varepsilon.$$

NO instance !

$$\therefore \cos^2\theta \leq \frac{1 + \cos\vartheta}{T+1} + \frac{T-1}{T+1} \leq 1 - \frac{1 - \sqrt{\varepsilon}}{T+1}$$

$$\sin^2\frac{\theta}{2} = \frac{1 - \sqrt{\cos^2\theta}}{2} \geq \frac{1 - \sqrt{\varepsilon}}{4(T+1)}$$

Taylor expand $\sqrt{1-x} \geq 1 - \frac{x}{2}$

Summary:

$$\lambda_{min}(A|_{\mathrm{supp}A}) \geq 1$$

$$\lambda_{min}(B|_{\mathrm{supp}B}) \geq \Omega(T^{-2})$$

$$\sin^2\frac{\theta}{2} \geq \frac{1-\sqrt{\varepsilon}}{4(T+1)}$$

Kitaev's Lemma $\Rightarrow$ for NO instance have

$$H = A + B \geq \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right)$$

i.e. have lower bound on λ_{min}:

$$\lambda_{min}(H) = \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right) \quad \text{as required.}$$

We have proven:

$$\text{YES} \Rightarrow \lambda_{min}(H) \leq \frac{\varepsilon}{T+1} =: \alpha$$

$$\text{NO} \Rightarrow \lambda_{min}(H) \geq \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right) =: \beta$$

Taking $\varepsilon = \Omega\left(\frac{1}{T^2}\right) \Rightarrow \beta - \alpha = \Omega\left(\frac{1}{T^3}\right)$

$\therefore$ Have proven reduction from QMA to Local Hamiltonian ...

... except that clock register has dimension $T+1$
$\longrightarrow$ H not on qudits with const. local dim. d.

Could split clock into $\log T$ qubits
= <u>binary</u> clock $(\mathbb{C}^T \cong (\mathbb{C}^2)^{\otimes \log T})$
$\longrightarrow$ local terms $\log T$-local, not k-local (k const).

(Example of general phenomenon: can always trade off locality against local dimension.)

Local Clock Construction

Problem with binary clock is that cannot tell what time it by looking at only a few of the bits $\longrightarrow$ non-local H.

Idea: use unary clock.

- Embedding $\iota: \mathbb{C}^{T+1} \hookrightarrow (\mathbb{C}^2)^{\otimes T}$

$$\mathbb{C}^{T+1} \longmapsto \mathcal{L} \subset (\mathbb{C}^2)^{\otimes T}$$

$$|t\rangle \longrightarrow |1\rangle^{\otimes t}\, |0\rangle^{\otimes T-t}$$

$$= |\underbrace{1, \cdots, 1}_{t}, \underbrace{0 \cdots 0}_{T-t}\rangle$$

(Label unary clock qubits $1, \ldots, T$)

- Replace clock terms:

$$|0\rangle\langle 0|_{cl} \longrightarrow |0\rangle\langle 0|_1$$

$$|T\rangle\langle T|_{cl} \longrightarrow |1\rangle\langle 1|_T$$

$$|t\rangle\langle t|_{cl} \longrightarrow |1\rangle\langle 1|_t \otimes |0\rangle\langle 0|_{t+1} \qquad t \neq 0, T$$

$$|t\rangle\langle t-1|_{cl} \longrightarrow |1\rangle\langle 1|_{t-1} \otimes |1\rangle\langle 0|_t \otimes |0\rangle\langle 0|_{t+1} \qquad t \neq 0, T$$

$$|1\rangle\langle 0|_{cl} \longrightarrow |1\rangle\langle 0|_1 \otimes |0\rangle\langle 0|_2$$

$$|T\rangle\langle T-1|_{cl} \longrightarrow |1\rangle\langle 1|_{T-1} \otimes |1\rangle\langle 0|_T$$

(Partially) defines a linear mapping on operators

[Super-formally:
Pad mapping to define action on remaining elems. $|t\rangle\langle s|$ & extend by linearity to
$\mathcal{K} : \mathcal{B}(\mathbb{C}^{T+1}) \longmapsto \mathcal{B}((\mathbb{C}^2)^{\otimes T})$.]

Under this mapping,

$$H = H_{in} + H_{prop} + H_{out} \longrightarrow H'$$

where H' acts on unary clock subspace

$\mathcal{L} = \mathrm{span}\{|1\rangle^{\otimes t}|0\rangle^{\otimes T-t}\} \subset (\mathbb{C}^2)^{\otimes T}$ in just the same way as H acts on original clock $\mathbb{C}^{T+1}$.

[Super-formally:

$$\begin{tikzcd}
\mathbb{C}^{T+1} \arrow[r, "H"] \arrow[d, "\iota"'] & \mathbb{C}^{T+1} \arrow[d, "\iota"] \\
(\mathbb{C}^2)^{\otimes T} \arrow[r, "\mathcal{K}(H)"'] & (\mathbb{C}^2)^{\otimes T}
\end{tikzcd}$$

]

<u>Exercise</u>: Prove this.

What goes wrong if we use
$|t\rangle\langle t-1|_{cl} \longrightarrow |1\rangle\langle 1|_{t-1} \otimes |1\rangle\langle 0|_t$?

<u>Problem</u>: What about "extra" part of Hilbert space? ($\mathcal{L} = (\mathbb{C}^2)^{\otimes T} \setminus \iota(\mathbb{C}^{T+1})$)

Add additional term to Hamiltonian:

$$H_{stab} = \mathbb{1} \otimes \sum_{t=1}^{T-1} |0\rangle\langle 0|_t \otimes |1\rangle\langle 1|_{t+1}$$

$\longrightarrow$ gives energy penalty to states <u>not</u> of form $|1\cdots,1\,0,\cdots,0\rangle$ (i.e. to all states in $\mathcal{L}^{\perp}$).

<u>Claim</u>

Same YES/NO-instance bounds for $H' + H_{stab}$ as for H.

Proof is largely a matter of formalising the blindingly obvious!

Proof

Note: $\ker H_{stab} = \mathcal{L} \supseteq \operatorname{supp} H'$

$\operatorname{supp} H_{stab} = \mathcal{L}^{\perp}$

$$H' = \begin{array}{c} \begin{pmatrix} \boxed{H} & 0 \\ 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \mathcal{L} \\ \mathcal{L}^{\perp} \end{array}, \quad H_{stab} = \begin{array}{c} \begin{pmatrix} 0 & 0 \\ 0 & \boxed{P} \end{pmatrix} \end{array} \begin{array}{l} \mathcal{L} \\ \mathcal{L}^{\perp} \end{array}$$

YES instance:

$$\lambda_{min}(H' + H_{stab})$$

$$= \min\left[\lambda_{min}(H'|_{\mathcal{L}}),\ \lambda_{min}(H_{stab}|_{\mathcal{L}^{\perp}})\right]$$

by above Note

$$\leq \lambda_{min}(H'|_{\mathcal{L}}) = \lambda_{min}(H)$$

$$\leq \frac{\varepsilon}{T+1} \quad \text{as before.}$$

NO instance:

Note: subspace $\mathcal{L} \subset (\mathbb{C}^2)^{\otimes T}$ invariant under both H', H_{stab}

(i.e. $\forall |\psi\rangle \in \mathcal{L}$: $H'|\psi\rangle, H_{stab}|\psi\rangle \in \mathcal{L}$).

$\Rightarrow H' + H_{stab}$ decomposes as

$$H' = (H'|_{\mathcal{L}} + H_{stab}|_{\mathcal{L}}) \oplus (H'|_{\mathcal{L}^\perp} + H_{stab}|_{\mathcal{L}^\perp})$$

$$\begin{pmatrix} \boxed{H} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \boxed{P} \end{pmatrix} = \begin{pmatrix} \boxed{H} & \\ & \boxed{P} \end{pmatrix}$$

Now

- $(H'|_{\mathcal{L}} + H_{stab}|_{\mathcal{L}}) = H'|_{\mathcal{L}} + 0 \mathrel{\hat{=}} H \geqslant \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right)$.

  from previously

- $(H'|_{\mathcal{L}^\perp} + H_{stab}|_{\mathcal{L}^\perp}) = 0 + H_{stab}|_{\mathcal{L}^\perp} = H_{stab}|_{\operatorname{supp} H_{stab}}$

$$= \left(\sum_{t=1}^{T} \mathbb{1} \otimes \Pi_t^{(0)} \otimes \Pi_{t+1}^{(1)}\right)\Big|_{\operatorname{supp}} \geqslant 1.$$

$\therefore H' + H_{stab} \geqslant \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right)$ as before. $\square$

Overall Hamiltonian is:

$$H' + H_{stab} = H'_{in} + \sum_t H'_t + H'_{out} + H_{stab}$$

$$H'_{in} = \left(\Pi_1^{(1)} + \sum_{j \in A} \Pi_j^{(1)} \right) \otimes |0\rangle\langle 0|_1 \qquad \text{2-local}$$

$$H'_{out} = \Pi_1^{(0)} \otimes |1\rangle\langle 1|_1 \qquad \text{2-local}$$

$$H'_t = \frac{1}{2} \mathbb{1} \otimes \Big(\overbrace{|1\rangle\langle 1|_t \otimes |0\rangle\langle 0|_{t+1}}^{\text{2-local}} + \overbrace{|1\rangle\langle 1|_{t+1} \otimes |0\rangle\langle 0|_{t+2}}^{\text{2-local}} \Big)$$

$$- \frac{1}{2} \underbrace{U_{t+1} \otimes \left(|1\rangle\langle 1|_t \otimes |1\rangle\langle 0|_{t+1} \otimes |0\rangle\langle 0|_{t+2} \right)}_{\leq 5\text{-local}} - h.c.$$

(recall U_t is 1- or 2-qubit gate)

$$H_{stab} = \sum_{t=1}^{T-1} |0\rangle\langle 0|_t \otimes |1\rangle\langle 1|_{t+1} \qquad \text{2-local}$$

$\therefore$ Overall Hamiltonian $H' + H_{stab}$ is 5-local.

Have proven Local Hamiltonian problem is <u>QMA-hard</u> for k-local Hamiltonians with $k \geq 5$!

<u>Exercise</u>: Prove Local Ham. $\in$ QMA.

QED (Kitaev's thm)