Local Clock Construction

Problem with binary clock is that cannot tell what time it by looking at only a few of the bits $\longrightarrow$ non-local H.

Idea: use unary clock.

- Embedding $\iota : \mathbb{C}^{T+1} \hookrightarrow (\mathbb{C}^2)^{\otimes T}$

$$\mathbb{C}^{T+1} \longmapsto \mathcal{L} \subset (\mathbb{C}^2)^{\otimes T}$$

$$|t\rangle \longrightarrow |1\rangle^{\otimes t} |0\rangle^{\otimes T-t}$$

$$= |\underbrace{1, \cdots, 1}_{t}, \underbrace{0 \cdots 0}_{T-t}\rangle$$

(Label unary clock qubits $1, \ldots, T$)

- Replace clock terms:

$$|0\rangle\langle 0|_{cl} \longrightarrow |0\rangle\langle 0|_1$$

$$|T\rangle\langle T|_{cl} \longrightarrow |1\rangle\langle 1|_T$$

$$|t\rangle\langle t|_{cl} \longrightarrow |1\rangle\langle 1|_t \otimes |0\rangle\langle 0|_{t+1} \qquad t \neq 0, T$$

$$|t\rangle\langle t-1|_{cl} \longrightarrow |1\rangle\langle 1|_{t-1} \otimes |1\rangle\langle 0|_t \otimes |0\rangle\langle 0|_{t+1} \qquad t \neq 0, T$$

$$|1\rangle\langle 0|_{cl} \longrightarrow |1\rangle\langle 0|_1 \otimes |0\rangle\langle 0|_2$$

$$|T\rangle\langle T-1|_{cl} \longrightarrow |1\rangle\langle 1|_{T-1} \otimes |1\rangle\langle 0|_T$$

(Partially) defines a linear mapping on operators

[Super-formally:
Pad mapping to define action on remaining elems. $\mapsto X s 1$ & extend by linearity to
$\mathcal{K} : \mathcal{B}(\mathbb{C}^{T+1}) \longmapsto \mathcal{B}((\mathbb{C}^2)^{\otimes T})$.]

Under this mapping,

$$H = H_{in} + H_{prop} + H_{out} \longrightarrow H'$$

where H' acts on unary clock subspace

$\mathcal{L} = \mathrm{span}\{|1\rangle^{\otimes t}|0\rangle^{\otimes T-t}\} \subset (\mathbb{C}^2)^{\otimes T}$ in just the same way as H acts on original clock $\mathbb{C}^{T+1}$.

[Super-formally:

$$\begin{tikzcd}
\mathbb{C}^{T+1} \arrow[r, "H"] \arrow[d, "\iota"'] & \mathbb{C}^{T+1} \arrow[d, "\iota"] \\
(\mathbb{C}^2)^{\otimes T} \arrow[r, "\mathcal{K}(H)"'] & (\mathbb{C}^2)^{\otimes T}
\end{tikzcd}$$

]

<u>Exercise</u>: Prove this.

What goes wrong if we use

$$|t\rangle\langle t-1|_{cl} \longrightarrow |1\rangle\langle 1|_{t-1} \otimes |1\rangle\langle 0|_t \quad ?$$

Problem: What about "extra" part of Hilbert space? ($\mathcal{L} = (\mathbb{C}^2)^{\otimes T} \setminus \iota(\mathbb{C}^{T+1})$)

Add additional term to Hamiltonian:

$$H_{stab} = \mathbb{1} \otimes \sum_{t=1}^{T-1} |0\rangle\langle 0|_t \otimes |1\rangle\langle 1|_{t+1}$$

$\longrightarrow$ gives energy penalty to states <u>not</u> of form $|1\cdots,1\,0,\cdots,0\rangle$ (i.e. to all states in $\mathcal{L}^\perp$).

Claim

Same YES/NO-instance bounds for $H' + H_{stab}$ as for H.

Proof is largely a matter of formalising the blindingly obvious!

Proof

Note: $\ker H_{stab} = \mathcal{L} \supseteq \operatorname{supp} H'$

$\operatorname{supp} H_{stab} = \mathcal{L}^{\perp}$

$$H' = \begin{array}{cc} \begin{pmatrix} \boxed{H} & 0 \\ 0 & 0 \end{pmatrix} & \begin{array}{c} \mathcal{L} \\ \mathcal{L}^{\perp} \end{array} \end{array}, \quad H_{stab} = \begin{array}{cc} \begin{pmatrix} 0 & 0 \\ 0 & \boxed{P} \end{pmatrix} & \begin{array}{c} \mathcal{L} \\ \mathcal{L}^{\perp} \end{array} \end{array}$$

YES instance:

$$\lambda_{min}(H' + H_{stab})$$

$$= \min \left[\lambda_{min}(H'|_{\mathcal{L}}), \ \lambda_{min}(H_{stab}|_{\mathcal{L}^{\perp}}) \right]$$

by above Note

$$\leq \lambda_{min}(H'|_{\mathcal{L}}) = \lambda_{min}(H)$$

$$\leq \frac{\varepsilon}{T+1} \quad \text{as before.}$$

NO instance:

Note: subspace $\mathcal{L} \subset (\mathbb{C}^2)^{\otimes T}$ invariant under both H', H_{stab}

(i.e. $\forall |\psi\rangle \in \mathcal{L}$: $H'|\psi\rangle$, $H_{stab}|\psi\rangle \in \mathcal{L}$).

$\Rightarrow$ $H' + H_{stab}$ decomposes as

$$H' = (H'|_{\mathcal{L}} + H_{stab}|_{\mathcal{L}}) \oplus (H'|_{\mathcal{L}^\perp} + H_{stab}|_{\mathcal{L}^\perp})$$

$$\begin{pmatrix} \boxed{H} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \boxed{P} \end{pmatrix} = \begin{pmatrix} \boxed{H} & \\ & \boxed{P} \end{pmatrix}$$

Now

- $(H'|_{\mathcal{L}} + H_{stab}|_{\mathcal{L}}) = H'|_{\mathcal{L}} + 0 \mathrel{\hat{=}} H \geqslant \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right)$.

  from previously

- $(H'|_{\mathcal{L}^\perp} + H_{stab}|_{\mathcal{L}^\perp}) = 0 + H_{stab}|_{\mathcal{L}^\perp} = H_{stab}|_{\operatorname{supp} H_{stab}}$

$$= \left(\sum_{t=1}^{T} \mathbb{1} \otimes \Pi_t^{(0)} \otimes \Pi_{t+1}^{(1)}\right)\Big|_{\operatorname{supp}} \geqslant 1.$$

$\therefore$ $H' + H_{stab} \geqslant \Omega\left(\frac{1-\sqrt{\varepsilon}}{T^3}\right)$ as before.

$\square$

Overall Hamiltonian is:

$$H' + H_{stab} = H'_{in} + \sum_t H'_t + H'_{out} + H_{stab}$$

$$H'_{in} = \left(\Pi_1^{(1)} + \sum_{j \in A} \Pi_j^{(1)}\right) \otimes |0\rangle\langle 0|_1 \qquad \text{2-local}$$

$$H'_{out} = \Pi_1^{(0)} \otimes |1\rangle\langle 1|_1 \qquad \text{2-local}$$

$$H'_t = \frac{1}{2} \mathbb{1} \otimes \Big(\overbrace{|1\rangle\langle 1|_t \otimes |0\rangle\langle 0|_{t+1}}^{\text{2-local}} + \overbrace{|1\rangle\langle 1|_{t+1} \otimes |0\rangle\langle 0|_{t+2}}^{\text{2-local}}$$

$$\underbrace{- \frac{1}{2} U_{t+1} \otimes \left(|1\rangle\langle 1|_t \otimes |1\rangle\langle 0|_{t+1} \otimes |0\rangle\langle 0|_{t+2} \right)}_{\le 5\text{-local}} - h.c.$$

(recall U_t is 1- or 2-qubit gate)

$$H_{stab} = \sum_{t=1}^{T-1} |0\rangle\langle 0|_t \otimes |1\rangle\langle 1|_{t+1} \qquad \text{2-local}$$

$\therefore$ Overall Hamiltonian $H' + H_{stab}$ is 5-local.

Have proven Local Hamiltonian problem is <u>QMA-hard</u> for k-local Hamiltonians with $k \geq 5$!

<u>Exercise</u>: Prove Local Ham. $\in$ QMA.

QED (Kitaev's thm)