Example Sheet 1
Hamiltonian Complexity

A. Complexity Theory

1. Show that if you can solve the Factoring decision problem in time t, you can find the factors of an n-digit number in time $O(tn)$. Give a decision variant of the Travelling Salesman problem, and prove a similar relationship to finding the lowest-weight Hamiltonian cycle in a weighted undirected graph.

2. Prove that the Factoring, Travelling Salesman, and SAT problems are all in the class NP.

3. Prove that the classes BQP and QMA are independent of the choice of YES/NO probabilities $1-1/\epsilon$ as long as $\epsilon = \Omega(1/\operatorname{poly}(n))$.

B. Hamiltonian Complexity

4. Assume that there exist local Hamiltonian terms H_t s.t. $\ker\left(\sum_t H_t\right) = \mathcal{L}_0$ where

$$\mathcal{L}_0 = \operatorname{span}\left\{ |\psi\rangle \quad : \quad \begin{array}{l} |\psi\rangle = |\psi_0\rangle\,|\psi_1\rangle \dots |\psi_T\rangle \\ |\psi_0\rangle = |0\rangle_1\,|\varphi\rangle\,\underbrace{|0\rangle\dots|0\rangle}_{\text{ancillas}} \\ |\psi_t\rangle = U_t\,|\psi_{i-1}\rangle \end{array} \right\}, \tag{1}$$

and $\lambda_{\min}\left(\sum_t H_t|_{\mathcal{L}^\perp}\right) \geq 1$. Use this assumption to prove Kitaev's theorem using the naive approach.
(As shown in lectures and in Question 6, this assumption is false in general. However, Hamiltonian terms with the right properties do exist in certain special cases, cf. Question 5.)

5. (Cook-Levin theorem) A classical k-local Hamiltonian on n bits can be viewed as a quantum k-local Hamiltonian on n qubits whose local terms are diagonal in the computational basis (i.e. the product basis $\bigotimes_{i=1}^n |x_i\rangle \in (\mathbb{C}^2)^{\otimes n}$, $x_i \in \{0,1\}$).

   Using Question 4, or otherwise, prove that the Local Hamiltonian problem for classical Hamiltonians is NP-complete. Use this to prove the Cook-Levin theorem, which states that the SAT problem is NP-complete.

6. Prove that $\forall U : \nexists H_{12}$ s.t. the ground state subspace $\mathcal{L}_0(H_{12} \otimes \mathbb{1}_{34}) = \operatorname{span}\{|\psi\rangle_{13} \otimes (U \otimes \mathbb{1})\,|\psi\rangle_{34}\}$.

7. Prove that the Feynman Hamiltonian $H \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^{T+1})$, where

$$H = (\mathbb{1} - |\psi_0\rangle\langle\psi_0|) \otimes |0\rangle\langle 0| + \sum_{t=0}^{T} \Big(|\psi_t\rangle\langle\psi_t| \otimes |t\rangle\langle t| + |\psi_{t+1}\rangle\langle\psi_{t+1}| \otimes |t+1\rangle\langle t+1| - |\psi_{t+1}\rangle\langle\psi_t| \otimes |t+1\rangle\langle t| - |\psi_t\rangle\langle\psi_{t+1}| \otimes |t\rangle\langle t+1| \Big), \tag{2}$$

has the computational history superposition state

$$|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\psi_t\rangle\,|t\rangle \tag{3}$$

as its unique 0-energy ground state.