Exponential decay of correlations

Condensed matter physics folk-lore:
spectral gap $\Leftrightarrow$ exponential decay of correlations (with distance).

(In relativistic QM, this is Fredenhagen's Thm.)

Only problem is, this folk-lore is false! (in non-relativistic many-body QM).

There are examples of gapless many-body systems with exp. decay of correlations:
exp. decay $\not\Rightarrow$ spectral gap.

What about converse?

Widely believed, but not proven until 2004/2005 by Matt Hastings using Lieb-Robinson techniques.

Thm (Exponential clustering)

k-Local Hamiltonian H with:

- spectral gap $\Delta > 0$
- unique ground state $|\phi_0\rangle$
- Lieb-Robinson constants μ, s

A_X, B_Y operators on subsets of qudits X, Y.

Then

$$\langle\phi_0| A_X B_Y |\phi_0\rangle - \langle\phi_0|A_X|\phi_0\rangle\langle\phi_0| B_Y |\phi_0\rangle \leq O\left(\|A\|\,\|B\| \min(|X|,|Y|)\, e^{-\tilde{\mu}\, d(X,Y)}\right)$$

where $\tilde{\mu} = \dfrac{\mu}{1 + 4ks/\Delta}$.

We will prove this result rigorously shortly, but first we will spend a little time discussing the intuition behind the proof.

In proving that time-evolved observables can be approximated by observables with compact support, it was intuitively clear that Lieb-Robinson bounds should play a role (though it required an elegant mathematical trick to make the connection), as the result itself concerned <u>time-evolution</u>.

Exponential decay of correlations concerns <u>static</u> properties of the system in its ground state.
→ Where do Lieb-Robinson bounds come in?

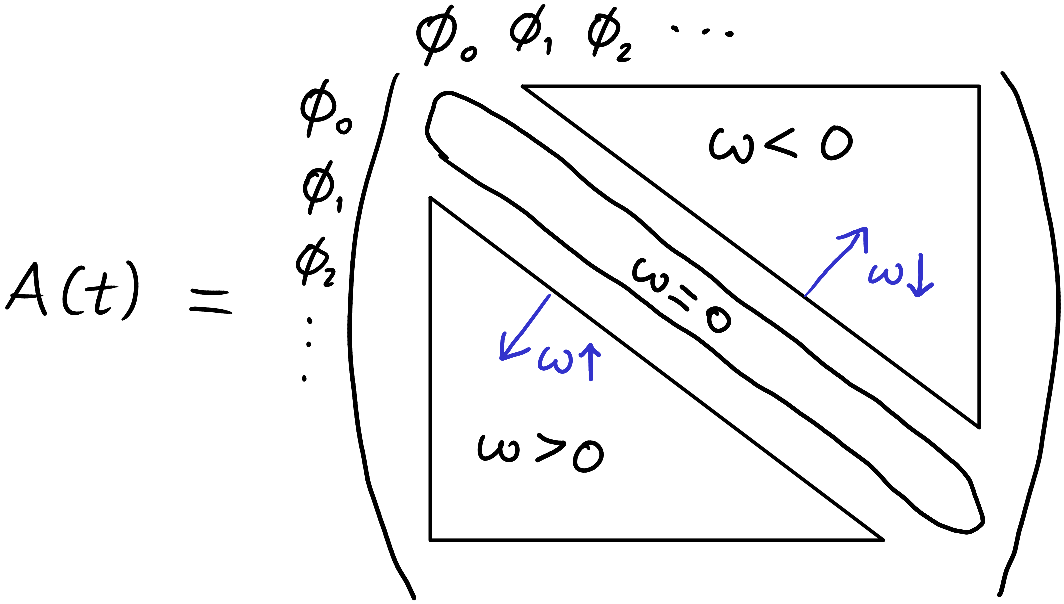
Consider matrix elements of an operator A in energy eigenbasis (i.e. eigenbasis of Hamiltonian):

$$A_{ij} = \langle \phi_i | A | \phi_j \rangle \quad , \qquad H|\phi_i\rangle = E_i |\phi_i\rangle$$

Matrix elements evolve as:

$$\begin{aligned} A_{ij}(t) &= \langle \phi_i | A(t) | \phi_j \rangle \\ &= \langle \phi_i | e^{iHt} A e^{-iHt} | \phi_j \rangle \\ &= A_{ij} \, e^{i \overbrace{(E_i - E_j)}^{} t} \end{aligned}$$

→ frequency ω_{ij}



Can "select" different "sectors" of matrix elements of A in energy eigenbasis using Fourier analysis of time-evolved operator $A(t)$:

1. Fourier transform $A(t)$
2. Apply frequency filter
3. Inverse Fourier transform
4. Evaluate result at $t = 0$

Let's see how this might work for ground state correlations...

Assume $\langle \phi_0 | A | \phi_0 \rangle = 0$.

(In fact, will see later that this is wlog.)

Then

$$\langle \phi_0 | AB | \phi_0 \rangle - \langle \phi_0 | A | \phi_0 \rangle \langle \phi_0 | B | \phi_0 \rangle$$

$$= \langle \phi_0 | AB | \phi_0 \rangle$$

$$= (1, 0, \cdots, 0) \begin{pmatrix} 0 & \text{———} \\ & A_{ij} \end{pmatrix} \begin{pmatrix} | & & \\ | & B_{ij} & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= (1, 0, \cdots, 0) \underbrace{\begin{pmatrix} 0 & \omega < 0 \\ 0 & \end{pmatrix}}_{A_{\omega<0}} \begin{pmatrix} & B_{ij} & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So only negative frequency sector of A is relevant for correlation function.

To select -ve frequency sector of A:

1. Fourier transform $A(t)$: $\hat{A}(\omega)$
2. Apply frequency filter: $\hat{A}(\omega)\,\Theta(\omega)$

   Step function $\Theta(\omega) = \begin{cases} 1 & \omega < 0 \\ 0 & \omega > 0 \end{cases}$

3. Inverse Fourier transform: $A(t) * \hat{\Theta}(t)$

$$= \int dt'\, A(t')\,\hat{\Theta}(t-t')$$

4. Evaluate result at $t=0$: $\int dt'\, A(t')\,\hat{\Theta}(-t')$

$$\longrightarrow \; A_{\omega<0} = \int dt\, A(t)\,\hat{\Theta}(-t)$$

How can we relate this to Lieb-Robinson?

Note

$$\langle \phi_0 | B\, A_{\omega<0} | \phi_0 \rangle$$

$$= (1, 0, \cdots, 0) \begin{pmatrix} & & \\ & B_{ij} & \\ & & \end{pmatrix} \begin{pmatrix} 0 & & \omega<0 \\ & \ddots & \\ 0 & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= 0$$

$$\Rightarrow \langle \phi_0 | A B | \phi_0 \rangle = \langle \phi_0 | [A_{\omega<0}, B] | \phi_0 \rangle .$$

$$= \int dt \, \langle \phi_0 | [A(t), B] | \phi_0 \rangle \, \hat{\Theta}(-t)$$

For small values of t, $[A(t), B]$ is small by L-R.

Problem: integral is dominated by large t component where L-R bound is trivial.

$\longrightarrow$ Introduce Gaussian envelope to suppress large t tail:

$$\int dt \, \langle \phi_0 | [A(t), B] | \phi_0 \rangle \, \hat{\Theta}(-t) \, e^{-\alpha t^2}$$

(compare with $A_{\omega<0}$, above)

Gaussian envelope of width $\sim 1/\alpha$

But now we are no longer exactly selecting -ve frequency component:

Fourier-transforming:

$$\mathcal{F}\left(\langle \phi_0 | [A(t), B] | \phi_0 \rangle \, \hat{\Theta}(-t) \, e^{-\alpha t^2} \right)$$

$$= \langle \phi_0 | [\tilde{A}(\omega), B] | \phi_0 \rangle \, \mathcal{F}\left(\hat{\Theta}(-t) \, e^{-\alpha t^2} \right)$$

$$\simeq \langle \phi_0 | [\tilde{A}(\omega), B] | \phi_0 \rangle \left(\Theta(-\omega) * e^{-\omega^2/4\alpha} \right)$$

convolution of step function with Gaussian of width $\sim \alpha$

To control integral of commutator, want fast-decaying Gaussian (α big).

For good approximation to -ve frequency filter, want fast-decaying Fourier transform, i.e. want flat Gaussian (α small).

$\longrightarrow$ Trade off control of long-time tail of integral against accuracy of frequency filter to get optimal bound on correlation function.

Now we make this intuition rigorous...

Note Fourier Transform convention in operation:

$$\hat{f}(\omega) = \mathcal{F}[f(t)] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt$$

$$f(t) = \mathcal{F}^{-1}[\hat{f}(\omega)] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega)\, d\omega$$

so

$$\mathcal{F}[\mathcal{F}[f(t)]] = f(-t).$$

<u>Thm</u> (Exponential clustering – Hastings)

k-local Hamiltonian H with:

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Then

$$\langle \phi_0 | A_X B_Y | \phi_0 \rangle - \langle \phi_0 | A_X | \phi_0 \rangle \langle \phi_0 | B_Y | \phi_0 \rangle$$

$$\leq O\left(\|A\| \, \|B\| \min(|X|, |Y|) \, e^{-\tilde{\mu} d(X,Y)} \right)$$

where $\tilde{\mu} = \dfrac{\mu}{1 + 4ks/\Delta}$.

For proof, will need following:

<u>Lemma</u>

For any $\alpha > 0$, $E \in \mathbb{R}$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Theta}(-t) e^{-iEt} e^{-\alpha t^2} dt = \begin{cases} 1 + O(e^{-\Delta^2/4\alpha}) & E \geq \Delta \\ O(e^{-\Delta^2/4\alpha}) & E \leq -\Delta \end{cases}$$

where $\hat{\Theta}(t) = \sqrt{\frac{\pi}{2}} \left(\delta(t) + \frac{i}{\pi t} \right)$ is the Fourier transform of the step function

$$\Theta(\omega) = \begin{cases} 1 & \omega \leq 0 \\ 0 & \omega > 0 . \end{cases}$$

We defer the proof of this Lemma to later.

Proof (of Thm)

Wlog can take $\langle \phi | A | \phi \rangle = \langle \phi | B | \phi \rangle = 0$.
(Let $A' = A - \alpha \mathbb{1}$, $B' = B - \beta \mathbb{1}$
where $\alpha = \langle \phi | A | \phi \rangle$, $\beta = \langle \phi | B | \phi \rangle$.
$\langle \phi | A' B' | \phi \rangle = \langle \phi | A B | \phi \rangle - \langle \phi | A | \phi \rangle \langle \phi | B | \phi \rangle$.)

Define

$$\tilde{A}_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} A_x(t)\, dt .$$

$$\langle \phi | A_x B_y | \phi \rangle - \cancel{\langle \phi | A_x | \phi \rangle \langle \phi | B_y | \phi \rangle}^{=0}$$

$$= \langle \phi | [\tilde{A}_x, B_y] | \phi \rangle + \langle \phi | A_x B_y | \phi \rangle - \langle \phi | [\tilde{A}_x, B_y] | \phi \rangle$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} \langle \phi | [A_x(t), B_y] | \phi \rangle \qquad (1)$$

$$- \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} \langle \phi | [A_x(t), B_y] | \phi \rangle - \langle \phi | A_x B_y | \phi \rangle \right) \qquad (2)$$

Term (1)

Lieb-Robinson bound controls integrand for small t, Gaussian tail controls large t:

$$\left| \int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} \langle \phi_0 | [A_X(t), B_Y] | \phi_0 \rangle \, dt \right|$$

$$= \left| \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \left(\delta(-t) - \frac{i}{\pi t} \right) e^{-\alpha t^2} \langle \phi_0 | [A_X(t), B_Y] | \phi_0 \rangle \, dt \right|$$

$$\leq \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} \left(\delta(t) + \frac{1}{\pi t} \right) e^{-\alpha t^2} \, \| [A_X(t), B_Y] \| \, dt$$

$$\leq \sqrt{\frac{\pi}{2}} \int_{|t| \leq cd} \left(\delta(t) + \frac{1}{\pi t} \right) \| [A_X(t), B_Y] \| \, dt + \sqrt{2\pi} \, \|A\| \, \|B\| \int_{|t| \geq cd} e^{-\alpha t^2} dt$$

where $d = d(X, Y)$

$$\leq \sqrt{\frac{\pi}{2}} \|A\| \cdot \|B\| \min(|X|, |Y|) \, e^{-\mu d} \int_{|t| \leq cd} \left(\delta(t) + \frac{1}{\pi t} \right) \left(e^{2ks t} - 1 \right) dt$$

$$+ \sqrt{2\pi} \, \|A\| \, \|B\| \, \frac{2\, e^{-\alpha c^2 d^2}}{\sqrt{\alpha}\, c d}$$

using Lieb-Robinson for 1st term,
& simple Gaussian tail bound for 2nd term:

$$\int_a^{\infty} e^{-x^2/2} dx < \int_a^{\infty} \frac{x}{a} e^{-x^2/2} dx = \frac{e^{-a^2}}{a}.$$

$$= O\left(\|A\|\,\|B\|\left(\min(|X|,|Y|)\, e^{-\mu d} e^{2kscd} + e^{-\alpha c^2 d^2}\right)\right)$$

using $\displaystyle\int_{-a}^{a} \frac{e^x - 1}{x}\, dx < \int_{-a}^{a} \frac{x e^x}{x}\, dx < e^a$

$$\leq O\left(\|A\|\,\|B\|\,\min(|X|,|Y|)\, e^{-d(\mu + c[\Delta/2 - 2ks)]}\right)$$

choosing $\alpha = \dfrac{\Delta}{2cd}$

$$= O\left(\|A\|\,\|B\|\,\min(|X|,|Y|)\, e^{-\tilde{\mu} d}\right)$$

where $\tilde{\mu} = \dfrac{\mu}{1 + 4ks/\Delta}$

choosing $c = \dfrac{\mu}{2ks + \Delta/2}$.

Term (2)

Bound using Lemma:

Let $P_0 := |\phi_0\rangle\langle\phi_0|$,
$|\phi_n\rangle$ be energy E_n eigstate of H
(i.e. $H|\phi_n\rangle = E_n|\phi_n\rangle$).

Note $\langle\phi_0|A P_0 B|\phi_0\rangle = 0$ by assumption.

We have

$$
\begin{aligned}
&\langle\phi|[A(t), B]|\phi\rangle \\
&= \langle\phi_0|A(t)(\mathbb{1} - P_0)B|\phi_0\rangle \\
&\quad - \langle\phi_0|B(\mathbb{1} - P_0)A(t)|\phi_0\rangle \\
&= \sum_{n\neq 0}\Big(\langle\phi_0|e^{iHt} A e^{-iHt}|\phi_n\rangle\langle\phi_n|B|\phi_0\rangle \\
&\qquad - \langle\phi_0|B|\phi_n\rangle\langle\phi_n|e^{iHt} A e^{-iHt}|\phi_0\rangle\Big) \\
&= \sum_{n\neq 0}\Big(\langle\phi_0|A|\phi_n\rangle\langle\phi_n|B|\phi_0\rangle e^{-it(E_n - E_0)} \\
&\qquad - \langle\phi_0|B|\phi_n\rangle\langle\phi_n|A|\phi_0\rangle e^{it(E_n - E_0)}\Big) \quad (*)
\end{aligned}
$$

Thus

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} \langle \phi | [A_X(t),\, B_Y] | \phi \rangle$$

$$= \sum_{n\neq 0} \Bigg(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} e^{-it\underbrace{(E_n - E_0)}_{\geq \Delta}} \langle \phi_0 | A | \phi_n \rangle\langle \phi_n | B | \phi_0 \rangle$$

$$- \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\Theta}(-t)\, e^{-\alpha t^2} e^{-it\underbrace{(E_0 - E_n)}_{\leq -\Delta}} \langle \phi_0 | B | \phi_n \rangle\langle \phi_n | A | \phi_0 \rangle \Bigg)$$

$$= \sum_{n\neq 0} \Big(\langle \phi_0 | A | \phi_n \rangle\langle \phi_n | B | \phi_0 \rangle \left[1 + O(e^{-\Delta^2/4\alpha})\right]$$

$$- \langle \phi_0 | B | \phi_n \rangle\langle \phi_n | A | \phi_0 \rangle \left[O(e^{-\Delta^2/4\alpha})\right] \Big)$$

using Lemma & <u>assumption on spectral gap</u>

$$= \langle \phi_0 | A\, (\mathbb{1} - P_0)\, B | \phi_0 \rangle$$

$$+ O\left(\|A\| \cdot \|B\| \cdot e^{-\Delta^2/4\alpha} \right)$$

$$= \langle \phi_0 | A B | \phi_0 \rangle + O\left(\|A\|\, \|B\|\, e^{-\Delta^2/4\alpha} \right)$$

recall $\langle \phi_0 | A | \phi_0 \rangle = \langle \phi_0 | B | \phi_0 \rangle = 0$
by assumption

Putting bounds together, we have

$$\langle \phi_0 | A_X B_Y | \phi_0 \rangle - \langle \phi_0 | A_X | \phi_0 \rangle \langle \phi_0 | B_Y | \phi_0 \rangle$$

$$\leq O\left(\|A\| \, \|B\| \min(|X|, |Y|) e^{-\tilde{\mu} d(X,Y)} \right)$$

$$- \left(\cancel{\langle \phi_0 | A_X B_Y | \phi_0 \rangle} + O\left(\|A\| \cdot \|B\| \, e^{-\Delta^2/4\alpha} \right) - \cancel{\langle \phi_0 | A_X B_Y | \phi_0 \rangle} \right)$$

$$= O\left(\|A\| \, \|B\| \min(|X|, |Y|) \, e^{-\tilde{\mu} d(X,Y)} \right)$$

recalling that we earlier chose

$$\alpha = \frac{\Delta}{2cd} \quad \& \quad c = \frac{\mu}{2ks + \Delta/2} \qquad \square$$

Lemma

For any $\alpha > 0$, $E \in \mathbb{R}$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\Theta}(-t) e^{-iEt} e^{-\alpha t^2} dt = \begin{cases} 1 + O(e^{-\Delta^2/4\alpha}) & E \geq \Delta \\ O(e^{-\Delta^2/4\alpha}) & E \leq -\Delta \end{cases}$$

where $\hat{\Theta}(t) = \sqrt{\frac{\pi}{2}}\left(\delta(t) + \frac{i}{\pi t}\right)$ is the Fourier transform of the step function

$$\Theta(\omega) = \begin{cases} 1 & \omega \leq 0 \\ 0 & \omega > 0 . \end{cases}$$

Proof

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{-iEt}\, \hat{\Theta}(-t) \underbrace{e^{-\alpha t^2}}_{G_\alpha(t)} dt \quad = \mathcal{F}[\hat{\Theta}(-t) G_\alpha(t)]$$

$$\overset{\shortparallel}{\hat{\Theta}(-t)\, G_\alpha(-t)}$$

$$= \frac{1}{\sqrt{2\pi}} [\Theta * \hat{G}_\alpha](-E) \qquad \mathcal{F}[\hat{f}(t)] = f(-\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \Theta(-E-\omega)\, \frac{1}{\sqrt{2\alpha}} e^{-\omega^2/4\alpha}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \Theta(\omega - E)\, \frac{1}{\sqrt{2\alpha}} e^{-\omega^2/4\alpha} \qquad \begin{aligned} \omega &\to -\omega \\ d\omega &\to -d\omega \\ \pm\infty &\to \mp\infty \end{aligned}$$

$$= \frac{1}{2\sqrt{\alpha\pi}} \int_{-\infty}^{E} d\omega\, e^{-\omega^2/4\alpha}$$

$$= \begin{cases} \frac{1}{2\sqrt{\alpha\pi}} \left(\int_{-\infty}^{\infty} d\omega \, e^{-\omega^2/4\alpha} - \int_{E}^{\infty} d\omega \, e^{-\omega^2/4\alpha} \right) & E \geqslant \Delta \\ \frac{1}{2\sqrt{\alpha\pi}} \int_{-\infty}^{E} d\omega \, e^{-\omega^2/4\alpha} & E \leq -\Delta \end{cases}$$

$$= \begin{cases} 1 + O\left(e^{-\Delta^2/4\alpha}\right) & E \geqslant \Delta \\ O\left(e^{-\Delta^2/4\alpha}\right) & E \leq -\Delta \end{cases}$$

using $\int_{E}^{\infty} d\omega \, e^{-\omega^2/4\alpha} \leq \int_{E}^{\infty} d\omega \, \frac{\omega}{\Delta} \, e^{-\omega^2/4\alpha} \qquad E \geq \Delta$

$$\leq \frac{2\alpha}{\Delta} \, e^{-\Delta^2/4\alpha}$$

$\square$